THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
FINAL EXAM: MAT300 Vector Analysis
DATE: 8. December 2020, 09:00 – 13:00
THE EXAM CONSISTS OF EXERCISES 0, 1, 2, AND 3.
THE FINAL PAGE CONTAINS SEVERAL USEFUL FORMULAS.
EACH OF THE 10 PARTS 1a, 1b, 1c, 1d, 2a, 2b, 3a, 3b, 3c, 3d ARE WORTH EQUAL MARKS.
UNLESS SPECIFIED OTHERWISE, SHOW ALL OF YOUR WORKING.

EXERCISE 0

As a security precaution, some questions in this exam will ask you to use your *magic number*, M. Each student will have their own magic number M, and anywhere that you see the constant M in this exam, you must *immediately* replace it by your own magic number.
To compute your magic number M, you need to *add together the first and last digits of your candidate number* for this exam. For example, if your candidate number were 5167, then your magic number would be $M = 5 + 7 = 12$.

a) Compute your own individual magic number M now, by adding together the first and last digits of *your own candidate number* for this exam. Your magic number will be a value between 5 and 15. Write your magic number M at the start of your exam paper. Make sure to show your working.

EXERCISE 1

In this exercise you will need to use some of the trigonometric identities at the bottom of the formula sheet on the final page of the exam.

Consider the curve $\mathscr{C}_1$: $\mathbf{r}(t) = \frac{1}{2}\sin(2t)\,\mathbf{i} + \sin^2(t)\,\mathbf{j} + e^t\,\mathbf{k}$, $0 \le t \le \frac{\pi}{2}$.

a) Find a unit tangent vector to $\mathscr{C}_1$ at the point corresponding to $t = \frac{\pi}{4}$.

b) (i) Copy the following line integral into your exam paper, making sure to replace M by your magic number from Exercise 0.

$$\int_{\mathscr{C}_1} \frac{2Mz^2}{(1+z^2)^{\frac{3}{2}}}\,ds\,.$$

(ii) Compute the line integral that you copied down in (b)(i). Give the exact answer, NOT an approximate numerical value. The final integral can be computed by hand, but you are allowed to use digital tools. Except for the final integral, all working must be shown.

Consider the vector field given by

$$\mathbf{F}(x,y,z) = 2x\,\mathbf{j} + c(x^2 - y(1-y))\,\mathbf{k}\,,$$

where c is a constant defined by

$$c = \frac{3\pi}{2(1-e^{\frac{\pi}{2}})}\,.$$

c) Compute the line integral

$$\int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{r}\,.$$

Give the exact answer, NOT an approximate numerical value. The final integral can be computed by hand, but you are allowed to use digital tools. Except for the final integral, all working must be shown.

Let $\mathscr{C}_2$ be the oriented straight line segment starting at the point $(0,0,1)$ and ending at the point $(0,1,e^{\frac{\pi}{2}})$.

d) (i) Writing in English or Norwegian, carefully explain in your own words how you can construct a closed curve $\mathscr{C}$ using the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$. Make sure you justify your answer convincingly. You do *not* need to give an explicit parametrisation of $\mathscr{C}$.

(ii) SHOWING ALL WORKING, compute the line integral

$$\int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{r}\,.$$

(iii) Using your answers from part (c) and part (d)(ii), write down the value of the line integral

$$\oint_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}\,.$$

Based *only* on your value for this integral, can you say anything about whether the vector field $\mathbf{F}$ is conservative? Make sure to give a reason for your answer.

EXERCISE 2

Consider the transformation $u = y - x^3$, $v = x + y$, between the (x, y)-coordinates and the (u, v)-coordinates. Let R be the region in the xy-plane bounded by the four curves

$$y = x^3 + 1, \quad y = x^3 + 2, \quad y = M - x, \quad \text{and} \quad x = 0\,.$$

a) (i) Copy into your exam paper the equations of the four curves given above, making sure to replace M by your magic number from Exercise 0.

(ii) Sketch the region R in the xy-plane that is bounded by the curves you wrote down in (a)(i), and sketch the region S in the uv-plane that corresponds to R under the given coordinate transformation. Make sure to clearly label all lines and curves that you draw in both the xy-plane and the uv-plane.

b) (i) Calculate the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$. Leave your answer in terms of the variables x and y.

(ii) SHOWING ALL WORKING, use the change of coordinates given above to compute the double integral

$$\iint_R (3x^2 + 1)\left(\frac{1}{(y - x^3)^3} - \frac{3}{8}\right) dx\, dy\,.$$

EXERCISE 3

Consider the vector field

$$\mathbf{F}(x,y,z) = (yz - xz)\,\mathbf{i} + (xz - yz)\,\mathbf{j} + (z + z^2 - 2x^2 - 2y^2)\,\mathbf{k}\,.$$

a) Compute $\nabla \cdot \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let the surface $\mathscr{S}_1$ be the part of the top sheet ($z > 0$) of the hyperboloid of two sheets defined by $x^2 + y^2 - z^2 = 1 - M^2$, inside of the cylinder $x^2 + y^2 = 1$.

Let the surface $\mathscr{S}_2$ be the disc in the xy-plane of radius 1, centred at the origin.

b) (i) Copy into your exam paper the defining equation of the surface $\mathscr{S}_1$ given above, making sure to replace M by your magic number from Exercise 0.

(ii) Using the defining equation of $\mathscr{S}_1$ that you wrote down in (b)(i), and SHOWING ALL YOUR WORKING, compute the flux of $\mathbf{F}$ upward through the surface $\mathscr{S}_1$,

$$\iint_{\mathscr{S}_1} \mathbf{F} \cdot d\mathbf{S}\,.$$

c) SHOWING ALL YOUR WORKING, compute the flux of $\mathbf{F}$ upward through the surface $\mathscr{S}_2$,

$$\iint_{\mathscr{S}_2} \mathbf{F} \cdot d\mathbf{S}\,.$$

Let the surface $\mathscr{S}_3$ be the cylinder defined by $x^2 + y^2 = 1$, $0 \leq z \leq M$, where M is your magic number from Exercise 0.

d) (i) Copy into your exam paper the definition of the surface $\mathscr{S}_3$ given above, making sure to replace M by your magic number from Exercise 0.

(ii) Using the definition of $\mathscr{S}_3$ that you wrote down in (d)(i), and SHOWING ALL YOUR WORKING, use the Divergence theorem to compute the flux of $\mathbf{F}$ out of the surface $\mathscr{S}_3$,

$$\iint_{\mathscr{S}_3} \mathbf{F} \cdot d\mathbf{S}\,.$$

Hint: The surfaces $\mathscr{S}_1, \mathscr{S}_2$, and $\mathscr{S}_3$ together form a closed surface $\mathscr{S}$ that is the boundary of a region $T \subset \mathbb{R}^3$. Apply the Divergence theorem to T, making sure to pay careful attention to the orientations of $\mathscr{S}_1, \mathscr{S}_2$, and $\mathscr{S}_3$.

If you have time, you may wish to check your result by computing the flux integral directly.

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \, dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\cdot\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S}$ parametrised by $\mathbf{r}(x,y)$, $(x,y) \in R$:

$$\iint_{\mathscr{S}} f \; dS = \iint_R f \left|\frac{\partial\mathbf{r}}{\partial x} \times \frac{\partial\mathbf{r}}{\partial y}\right| dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f \; dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f \; dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\cdot \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\cdot\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla\cdot\mathbf{F}\; dV = \oiint\limits_{\mathscr{S}} \mathbf{F}\cdot\hat{\mathbf{N}}\; dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{N}}\; dS = \oint\limits_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\cdot\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric identities: $\sin^2\theta + \cos^2\theta = 1$, $\quad \sin 2\theta = 2\sin\theta\cos\theta$,

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

EXERCISE 1

$$\mathcal{C}_1 : \underset{\sim}{r}(t) = \tfrac{1}{2}\sin(2t)\,\underline{i} + \sin^2(t)\,\underline{j} + e^t\,\underset{\sim}{k}$$

$$0 \le t \le \pi/2$$

(a) $\underline{r}'(t) = \cos(2t)\,\underline{i} + 2\sin(t)\cos t\,\underline{j} + e^t\,\underline{k}$

$= \cos(2t)\,\underline{i} + \sin(2t)\,\underline{j} + e^t\,\underline{k}$

$\underline{r}'(\pi/4) = \cos(\pi/2)\,\underline{i} + \sin(\pi/2)\,\underline{j} + e^{\pi/4}\,\underline{k}$

$= \underline{j} + e^{\pi/4}\,\underline{k}$

$|\underline{r}'(\pi/4)| = \sqrt{1 + (e^{\pi/4})^2} = \sqrt{1 + e^{\pi/2}}$

$$\underset{\sim}{\hat{T}} = \frac{\underline{r}'(\pi/4)}{|\underline{r}'(\pi/4)|} = \frac{1}{\sqrt{1+e^{\pi/2}}}\left(\underline{j} + e^{\pi/4}\,\underline{k}\right).$$

(b) $\underset{\sim}{r}'(t) = \cos(2t)\,\underline{i} + \sin(2t)\,\underline{j} + e^{t}\,\underline{k}$

$$|\underset{\sim}{r}'(t)| = \sqrt{\cos^2(2t) + \sin^2(2t) + e^{2t}}$$
$$= \sqrt{1 + e^{2t}}$$

$$\int_{e_1} \frac{2Mz^2}{(1+z^2)^{3/2}}\,ds = \int_0^{\pi/2} \frac{2M e^{2t}}{(1+e^{2t})^{3/2}} (1+e^{2t})^{1/2}\,dt$$

$$= \int_0^{\pi/2} \frac{2M e^{2t}}{1+e^{2t}}\,dt$$

$$= M \ln(1+e^{2t}) \Big|_0^{\pi/2}$$

$$= M\left[\ln(1+e^{\pi}) - \ln 2\right]$$

$$= \ln\left(\left(\frac{1+e^{\pi}}{2}\right)^{M}\right)$$

(c) $\underset{\sim}{F}(x,y,z) = 2x\underset{\sim}{j} + c\left(x^2 - y(1-y)\right)\underset{\sim}{k}$

where $c = \dfrac{3\pi}{2(1-e^{\pi/2})}$.

$\underset{\sim}{F}\left(\frac{1}{2}\sin(2t),\ \sin^2 t,\ e^t\right)$

$= \sin(2t)\underset{\sim}{j} + c\left(\frac{1}{4}\sin^2 2t - \sin^2 t(1-\sin^2 t)\right)\underset{\sim}{k}$

$= \sin(2t)\underset{\sim}{j} + c\left(\sin^2 t\cos^2 t - \sin^2 t\cos^2 t\right)\underset{\sim}{k}$

$= \sin(2t)\underset{\sim}{j}$

$\underset{\sim}{F}\cdot\dfrac{d\underset{\sim}{r}}{dt} = \left[\sin(2t)\underset{\sim}{j}\right]\cdot\left[\cos(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j} + e^t\underset{\sim}{k}\right]$

$= \sin^2(2t)$

$\displaystyle\int_{\mathcal{C}_1} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_0^{\pi/2} \sin^2(2t)\,dt$

$\cos 4t = 1 - 2\sin^2 2t$

$\sin^2 2t = \frac{1}{2}(1-\cos 4t)$

$\displaystyle = \int_0^{\pi/2} \frac{1}{2}(1-\cos 4t)\,dt$

$\displaystyle = \frac{t}{2} - \frac{1}{8}\sin 4t\,\Big|_0^{\pi/2} = \frac{\pi}{4}$

(d) $\mathcal{C}_2$: line segment from $(0,0,1)$ to $(0,1,e^{\pi/2})$

(i) Note that for $\mathcal{C}_1$, $\underset{\sim}{r}(0) = (0,0,1)$
and $\underset{\sim}{r}(\pi/2) = (0,1,e^{\pi/2})$,
so $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same initial and final points.
If we define $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$, i.e., the curve $\mathcal{C}_1$ from $(0,0,1)$ to $(0,1,e^{\pi/2})$, followed by $\mathcal{C}_2$ <u>in reverse</u> from $(0,1,e^{\pi/2})$ to $(0,0,1)$, then $\mathcal{C}$ is a closed curve starting and finishing at the point $(0,0,1)$.

(ii) e_2 : $\underset{\sim}{s}(t) = t\,\underline{j} + (1 + (e^{\pi/2} - 1)t)\,\underline{k}$
$t \in [0,1]$.

$$\underset{\sim}{s}'(t) = \underline{j} + (e^{\pi/2} - 1)\,\underline{k}$$

$$\underset{\sim}{F} \cdot \underset{\sim}{s}'(t) = 2x + c(x^2 - y(1-y))(e^{\pi/2} - 1)$$

$$= 0 + c(0 - t + t^2)(e^{\pi/2} - 1)$$

$$\int_{e_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = c(e^{\pi/2} - 1)\int_0^1 t^2 - t \, dt$$

$$= c(e^{\pi/2} - 1)\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1$$

$$= c(e^{\pi/2} - 1)\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{6}(1 - e^{\pi/2}) \cdot \frac{3\pi}{2(1 - e^{\pi/2})} = \frac{\pi}{4}$$

(iii)

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} = \int_{\mathcal{C}_1} \underset{\sim}{F} \cdot d\underline{r} - \int_{\mathcal{C}_2} \underset{\sim}{F} \cdot d\underline{r}$$

$$= \pi/4 - \pi/4 = 0.$$

Even though this closed loop integral is zero, we cannot conclude anything from this about whether $\underline{F}$ is conservative. Only if the integral of $\underset{\sim}{F}$ around <u>every</u> closed curve were zero, could we conclude that $\underset{\sim}{F}$ is conservative, since, from lectures,

$$\underset{\sim}{F} \text{ conservative} \Leftrightarrow \oint_{\mathcal{L}} \underset{\sim}{F} \cdot d\underline{r} = 0 \text{ for every closed curve } \mathcal{L}.$$

EXERCISE 2

(a) $u = y - x^3, \qquad v = x + y$

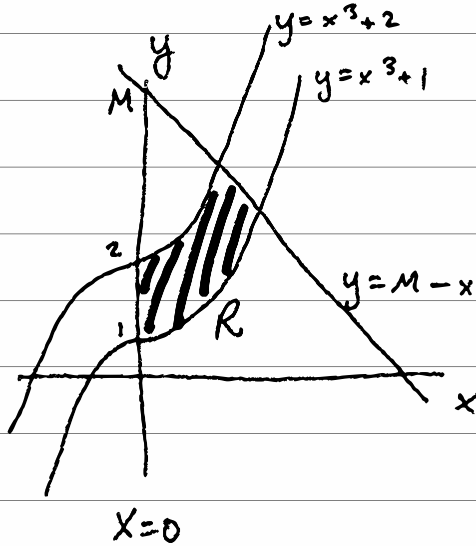
R: region bounded by curves

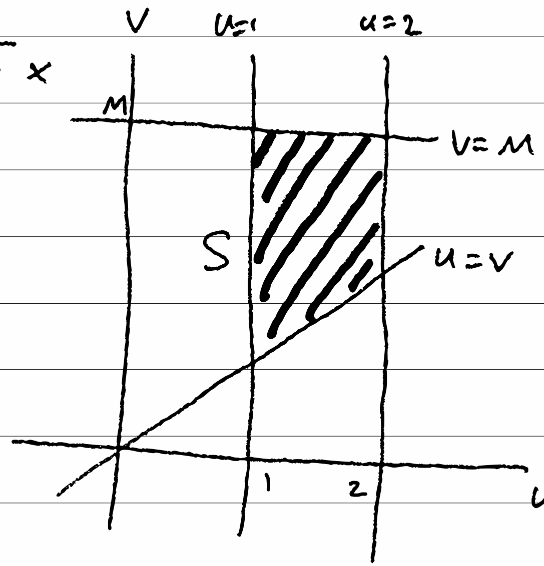
$y = x^3 + 1 \quad \Rightarrow \quad u = y - x^3 = 1$

$y = x^3 + 2 \quad \Rightarrow \quad u = y - x^3 = 2$

$y = M - x \quad \Rightarrow \quad v = x + y = M$

$x = 0 \quad \Rightarrow \quad u = y, \; v = y \Rightarrow u = v.$





(b)(i) $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -3x^2 & 1 \\ 1 & 1 \end{vmatrix} = -3x^2-1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = -\frac{1}{3x^2+1}.$$

(ii) $$\iint_R (3x^2+1)\left(\frac{1}{(y-x^3)^3} - \frac{3}{8}\right) dx\,dy$$

$$= \iint_S (3x^2+1)\left(\frac{1}{(y-x^3)^3} - \frac{3}{8}\right) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\frac{1}{3x^2+1}} du\,dv$$

$$= \iint_S \left(\frac{1}{u^3} - \frac{3}{8}\right) du\,dv$$

$$= \int_1^2 \int_u^M u^{-3} - 3/8 \, dv \, du$$

$$= \int_1^2 vu^{-3} - 3/8 \, v \Big|_u^M \, du$$

$$= \int_1^2 Mu^{-3} - \frac{3M}{8} - u^{-2} + \frac{3u}{8} \, du$$

$$= -\frac{Mu^{-2}}{2} - \frac{3Mu}{8} + u^{-1} + \frac{3u^2}{16} \Big|_1^2$$

$$= -\frac{M}{8} - \frac{3M}{4} + 1/2 + \frac{3}{4} - \left[-\frac{M}{2} - \frac{3M}{8} + 1 + \frac{3}{16} \right]$$

$$= M\left[-\frac{1}{8} - \frac{6}{8} + \frac{4}{8} + \frac{3}{8} \right] + \frac{8}{16} + \frac{12}{16} - \frac{16}{16} - \frac{3}{16} = \frac{1}{16}$$

EXERCISE 3

$$\underline{F}(x,y,z) = (yz - xz)\underset{\sim}{i} + (xz - yz)\underline{j} + (z + z^2 - 2x^2 - 2y^2)\underline{k}$$

(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = -z - z + 1 + 2z = 1$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ yz - xz & xz - yz & z + z^2 - 2x^2 - 2y^2 \end{vmatrix}$$

$$= \underset{\sim}{i}(-4y - x + y) + \underline{j}(y - x + 4x) + \underline{k}(z - z)$$

$$= \underset{\sim}{i}(-3y - x) + \underline{j}(y + 3x)$$

(b) $S_1 : x^2+y^2-z^2 = 1-M^2$, $z>0$, inside $x^2+y^2=1$.

(ii) let $G(x,y,z) = x^2+y^2-z^2$, then
$S_1 : G(x,y,z) = 1-M^2$, $z>0$, $x^2+y^2 \le 1$.
$\nabla G = 2x\underline{i} + 2y\underline{j} - 2z\underline{k}$, $\dfrac{\partial G}{\partial z} = -2z$

$$d\underset{\sim}{S} = \pm \frac{\nabla G}{\frac{\partial G}{\partial z}}\, dx\,dy = \pm\left(-\frac{x}{z}\underline{i} - \frac{y}{z}\underline{j} + \underline{k}\right) dx\,dy$$

We choose "+" to get flux upwards through S_1.

$$\underset{\sim}{F} \cdot d\underset{\sim}{S} = \Big[(yz-xz)\left(-\frac{x}{z}\right) + (xz-yz)\left(\frac{-y}{z}\right) + z + z^2 - 2x^2 - 2y^2\Big] dx\,dy$$

$$= \left[-xy + x^2 - xy + y^2 + z + z^2 - 2x^2 - 2y^2\right] dx\,dy$$

$$= \left[-2xy + z + z^2 - x^2 - y^2\right] dx\,dy$$

$$= -2xy + \sqrt{x^2+y^2+M^2-1} + M^2 - 1$$

The projection of S_1 onto the xy-plane is the disc S_2, $x^2+y^2 \leq 1$.

$$\iint_{S_1} \underset{\sim}{F} \cdot d\underline{S} = \iint_{S_2} \underbrace{-2xy}_{\text{vanishes due to symmetry}} + \sqrt{x^2+y^2+M^2-1} + \underbrace{M^2-1}_{\text{gives area of } S_2} \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{r^2+M^2-1}\, r\,dr\,d\theta + (M^2-1)\pi$$

now $\displaystyle\int_0^1 r\,(r^2+K)^{1/2}\,dr$

$$= \frac{2}{3}\cdot\frac{1}{2}\,(r^2+K)^{3/2}\Big|_0^1 = \frac{1}{3}\left[(K+1)^{3/2} - K^{3/2}\right]$$

so $\displaystyle\iint_{S_1} \underline{F}\cdot d\underline{S} = \frac{2\pi}{3}\left[M^3 - (M^2-1)^{3/2}\right] + (M^2-1)\pi$

M	$\frac{2\pi}{3}\left[M^3-(M^2-1)^{3/2}\right]+(M^2-1)\pi$
6	$\frac{2\pi}{3}\left[216-35^{3/2}\right]+35\pi$
7	$\frac{2\pi}{3}\left[343-48^{3/2}\right]+48\pi$
8	$\frac{2\pi}{3}\left[512-63^{3/2}\right]+63\pi$
9	$\frac{2\pi}{3}\left[729-80^{3/2}\right]+80\pi$
10	$\frac{2\pi}{3}\left[1000-99^{3/2}\right]+99\pi$
11	$\frac{2\pi}{3}\left[1331-120^{3/2}\right]+120\pi$
12	$\frac{2\pi}{3}\left[1728-143^{3/2}\right]+143\pi$
13	$\frac{2\pi}{3}\left[2197-168^{3/2}\right]+168\pi$
14	$\frac{2\pi}{3}\left[2744-195^{3/2}\right]+195\pi$

(c) For S_2, $d\underset{\sim}{S} = +\underset{\sim}{k}\, dx\, dy$ (upward flux)

$$\underset{\sim}{F} \cdot d\underset{\sim}{S} = (z + z^2 - 2x^2 - 2y^2)\, dx\, dy$$

$$= -2(x^2 + y^2)\, dx\, dy \qquad (z = 0 \text{ on } S_2).$$

$$\iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S} = -2 \int_0^{2\pi} \int_0^1 r^2 \, r\, dr\, d\theta$$

$$= -4\pi \cdot \frac{1}{4} = -\pi$$

(d) S_3: $x^2+y^2=1$, $0 \le z \le M$.

Look at the intersection of $x^2+y^2=1$ with $x^2+y^2-z^2=1-M^2$:

$$1-z^2=1-M^2$$
$$z^2=M^2$$
$$z=M \quad (\text{since } z>0)$$

So if we let T be the region bounded by the xy-plane, the cylinder $x^2+y^2=1$, and the top sheet of $x^2+y^2-z^2=1-M^2$, then the boundary surface S of T is $S=S_1-S_2+S_3$.

(where we take $-S_2$ to get the flux <u>out</u> of T, since S_2 was earlier taken to have the upwards pointing unit normal vector field.)

By the divergence theorem,

$$\iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \, dV = \oiint_S \underset{\sim}{F} \cdot d\underline{S}$$

$$= \iint_{S_1} \underline{F} \cdot d\underline{S} - \iint_{S_2} \underset{\sim}{F} \cdot d\underline{S} + \iint_{S_3} \underline{F} \cdot d\underline{S}$$

Now, $T: \ 0 \le \theta \le 2\pi, \ 0 \le r \le 1,$
$0 \le z \le \sqrt{r^2 + M^2 - 1}$

$$\iiint_T \underline{\nabla} \cdot \underline{F} \, dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{r^2+M^2-1}} 1 \, r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 r \sqrt{r^2 + M^2 - 1} \, dr = \frac{2\pi}{3} \left[M^3 - (M^2-1)^{3/2} \right]$$

$$\iint_{S_3} \underset{\sim}{F} \cdot d\underline{S} = \frac{2\pi}{3} \left[M^3 - (M^2-1)^{3/2} \right]$$
$$- \left(\frac{2\pi}{3} \left[M^3 - (M^2-1)^{3/2} \right] + (M^2-1)\pi \right) - \pi$$

$$= (1 - M^2)\pi - \pi$$

$$= - M^2\pi.$$