1 Assignment

In the mandatory exercises for this assignment, you will practice with the modeling of electrical circuits, with obtaining a state-space representation from a given differential equation, and the linearization of a nonlinear system around an equilibrium point. Two optional exercises will help you in understanding the motivation for the course ELE320 - Reguleringsteknikk, as well as to gain more insight into the linearization.

1.1 Modeling of electrical circuits (mandatory)

Question 1.1

Consider the electrical circuit in Fig. 1.1. Determine a model of how the input $v(t)$ affects the output $y(t) = v_{L_1(t)}$. What is the order of this circuit? You should observe a mismatch between the number of dynamical components and the order of the differential equation. Why do you think that this mismatch happens?

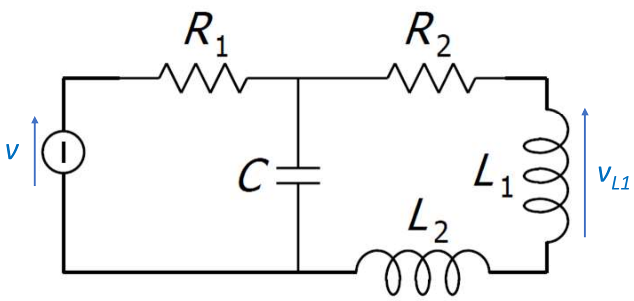

Figure 1.1: Electrical circuit (Exercise 1).

Solution: After defining positive verses for voltages and currents, for example as shown in Fig. 1.2, we can write down the components' equations:

$$\begin{cases} Resistor1 : V_1(t) = R_1 i_1(t) \\ Resistor2 : V_2(t) = R_2 i_2(t) \\ Capacitor : i_C(t) = C dV_C(t)/dt \\ Inductor1 : V_{L1}(t) = L_1 di_2(t)/dt \\ Inductor2 : V_{L2}(t) = L_2 di_2(t)/dt \end{cases} \tag{1.1}$$

and the Kirchhoff's laws:

$$\begin{cases} KVL1: v(t) - V_1(t) - V_C(t) = 0 \\ KVL2: V_C(t) - V_2(t) - V_{L1}(t) - V_{L2}(t) = 0 \\ KCL: i_1(t) - i_2(t) - i_C(t) = 0 \end{cases} \tag{1.2}$$

By replacing the components' equations (1.1) into (1.2), we obtain:

$$\begin{cases} KVL1: v(t) - R_1 i_1(t) - V_C(t) = 0 \\ KVL2: V_C(t) - R_2 i_2(t) - L_1 di_2(t)/dt - L_2 di_2(t)/dt = 0 \\ KCL: i_1(t) - i_2(t) - C dV_C(t)/dt = 0 \end{cases} \tag{1.3}$$

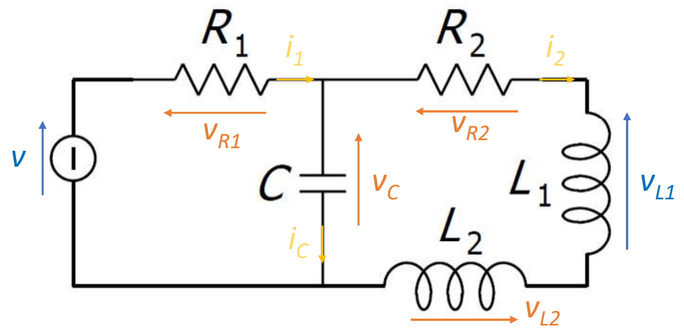

Figure 1.2: Electrical circuit (Exercise 1) - Solution.

We can replace $i_1(t)$ from KCL into KVL1 in (1.3) to obtain:

$$v(t) - R_1 i_2(t) - R_1 C dV_C(t)/dt - V_C(t) = 0 \quad \Rightarrow \quad i_2(t) = \frac{v(t) - C dV_C(t)/dt - V_C(t)}{R_1} \tag{1.4}$$

and replace this expression into KVL2:

$$V_C(t) - \frac{R_2}{R_1} v(t) + \frac{R_2 C}{R_1}\frac{dV_C(t)}{dt} + \frac{R_2}{R_1} V_C(t) - \frac{(L_1 + L_2)}{R_1}\left[\frac{dV(t)}{dt} - C\frac{d^2 V_C(t)}{dt^2} - \frac{dV_C(t)}{dt}\right] = 0 \tag{1.5}$$

We have obtained the second-order differential equation:

$$\frac{(L_1 + L_2)C}{R_1}\frac{d^2 V_C(t)}{dt^2} + \left[\frac{R_2 C}{R_1} + \frac{(L_1 + L_2)}{R_1}\right]\frac{dV_C(t)}{dt} + \left[1 + \frac{R_2}{R_1}\right] V_C(t) - \frac{R_2}{R_1} v(t) - \frac{(L_1 + L_2)}{R_1}\frac{dv(t)}{dt} = 0 \tag{1.6}$$

The equation for the output of interest is:

$$V_{L_1}(t) = L_1 \frac{di_2(t)}{dt} = \frac{L_1}{R_1}\frac{dv(t)}{dt} - \frac{L_1 C}{R_1}\frac{d^2 V_C(t)}{dt^2} - \frac{L_1}{R_1}\frac{dV_C(t)}{dt} \tag{1.7}$$

By looking at the obtained result, we might wonder about why a circuit with three dynamical components (the capacitor C and the inductors L_1, L_2) is described by a second-order differential equation. This is due to the fact that the series of L_1 and L_2 behaves as a single inductor with inductance $L_1 + L_2$, so we have only two dynamical components: the capacitor C and an inductor with inductance $L_1 + L_2$.

1.2 Obtaining a state-space representation (mandatory)

Question 1.2

The dynamic equation for a pendulum is given by:

$$ml\ddot{\theta}(t) = -mg\sin(\theta(t)) - kl\dot{\theta}(t) \tag{1.8}$$

where $l > 0$ is the pendulum's length, $m > 0$ is the mass, $k > 0$ is a friction parameter and $\theta(t)$ is the angle between the rod of the pendulum and the vertical axis. Choose appropriate state variables and write down the state equation.

Solution: The variable $\theta(t)$ appears differentiated twice, so we take: $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, from which we obtain:

$$\begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) = -\frac{g}{l}\sin(\theta(t)) - \frac{k}{m}\dot{\theta}(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{cases} \tag{1.9}$$

If we define the state vector $x(t) = [x_1(t), x_2(t)]^T$, we obtain:

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix} \tag{1.10}$$

1.3 Linearization around an equilibrium point (mandatory)

Question 1.3

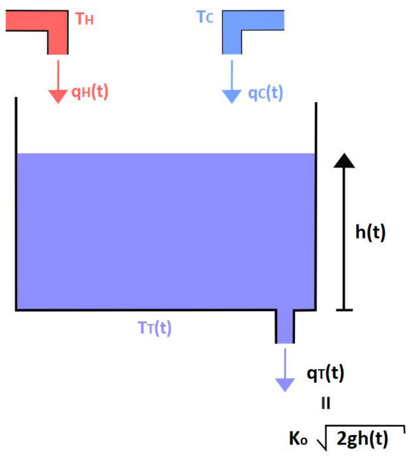

Figure 1.3: Tank with mixing flows of water.

We have found out that a tank where a flow $q_H(t)$ of hot water at temperature T_H and a flow $q_C(t)$ of cold water at temperature T_C get mixed (see Fig. 1.3) can be described by a compact state-space representation:

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.11}$$

with $x(t)$ and $u(t)$ defined as:

$$x(t) = \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} \tag{1.12}$$

where $h(t)$ is the water level in the tank and $T_T(t)$ denotes the temperature of the water inside the tank.

The function $f(x(t), u(t))$ is given by:

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix} = \begin{bmatrix} \frac{1}{A_T}\left(u_1 + u_2 - K_o\sqrt{2gx_1}\right) \\ \frac{1}{x_1 A_T}\left(u_1\left[T_C - x_2\right] + u_2\left[T_H - x_2\right]\right) \end{bmatrix} \tag{1.13}$$

and the equilibrium point is given by:

$$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{K_o\sqrt{2g\bar{h}}(T_H - \bar{T}_T)}{T_H - T_C} \\ \frac{K_o\sqrt{2g\bar{h}}(\bar{T}_T - T_C)}{T_H - T_C} \end{bmatrix} \quad \Rightarrow \quad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{h} \\ \bar{T}_T \end{bmatrix} \tag{1.14}$$

Assuming that $T_C = 20°$, $T_H = 90°$, $A_T = 3\,m^2$, $K_o = 0.035\,m^2$ and $g = 9.81\,m/s^2$ find the linearized systems:

$$\begin{cases} \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} = \begin{bmatrix} \bar{h} \\ \bar{T}_T \end{bmatrix} + \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix}, \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} = \begin{bmatrix} \bar{q}_C \\ \bar{q}_H \end{bmatrix} + \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}, \begin{bmatrix} h(0) \\ T_T(0) \end{bmatrix} = \begin{bmatrix} \bar{h} \\ \bar{T}_T \end{bmatrix} + \begin{bmatrix} \delta h(0) \\ \delta T_T(0) \end{bmatrix} \\ \begin{bmatrix} \delta \dot{h}(t) \\ \delta \dot{T}_T(t) \end{bmatrix} \approx A \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix} + B \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix} \end{cases} \tag{1.15}$$

that correspond to the following equilibrium points:

- $\left[\bar{h}, \bar{T}_T\right] = \left[1\,m, 30°\right]$
- $\left[\bar{h}, \bar{T}_T\right] = \left[1\,m, 70°\right]$
- $\left[\bar{h}, \bar{T}_T\right] = \left[4\,m, 30°\right]$
- $\left[\bar{h}, \bar{T}_T\right] = \left[4\,m, 70°\right]$

(Note: You should replace appropriate numbers/matrices in $\bar{h}$, $\bar{T}_T$, $\bar{q}_C$, $\bar{q}_H$, A, B. You could write a short MATLAB script that helps you with the calculations.)

Solution: Let's linearize the nonlinear system around the equilibrium point. To do so, we need to calculate $\bar{u}_1$ and $\bar{u}_2$ using Eq. (1.14) and the necessary partial derivatives to fill the matrices A

and B. We can calculate the following:

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{\partial}{\partial x_1}\left\{\frac{1}{A_T}\left(u_1+u_2-K_o\sqrt{2gx_1}\right)\right\} = -\frac{K_o}{A_T}\frac{\partial}{\partial x_1}\left\{\sqrt{2gx_1}\right\} \\
&= -\frac{K_o}{A_T}\frac{1}{2\sqrt{2gx_1}}\frac{\partial}{\partial x_1}\{2gx_1\} = -\frac{K_o}{A_T}\frac{g}{\sqrt{2gx_1}} = -\frac{K_o}{A_T}\sqrt{\frac{g}{2x_1}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_2} &= \frac{\partial}{\partial x_2}\left\{\frac{1}{A_T}\left(u_1+u_2-K_o\sqrt{2gx_1}\right)\right\} = 0 \\
\frac{\partial f_1}{\partial u_1} &= \frac{\partial}{\partial u_1}\left\{\frac{1}{A_T}\left(u_1+u_2-K_o\sqrt{2gx_1}\right)\right\} = \frac{1}{A_T} \\
\frac{\partial f_1}{\partial u_2} &= \frac{\partial}{\partial u_2}\left\{\frac{1}{A_T}\left(u_1+u_2-K_o\sqrt{2gx_1}\right)\right\} = \frac{1}{A_T}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_2}{\partial x_1} &= \frac{\partial}{\partial x_1}\left\{\frac{1}{x_1A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\right\} = \frac{1}{A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\frac{\partial}{\partial x_1}\left\{\frac{1}{x_1}\right\} \\
&= -\frac{1}{A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\frac{1}{x_1^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_2}{\partial x_2} &= \frac{\partial}{\partial x_2}\left\{\frac{1}{x_1A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\right\} = \frac{1}{x_1A_T}\left(u_1\frac{\partial}{\partial x_2}\{T_C-x_2\}+u_2\frac{\partial}{\partial x_2}\{T_H-x_2\}\right) \\
&= -\frac{1}{x_1A_T}\left(u_1+u_2\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_2}{\partial u_1} &= \frac{\partial}{\partial u_1}\left\{\frac{1}{x_1A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\right\} = \frac{1}{x_1A_T}\left(T_C-x_2\right) \\
\frac{\partial f_2}{\partial u_2} &= \frac{\partial}{\partial u_2}\left\{\frac{1}{x_1A_T}\left(u_1\left[T_C-x_2\right]+u_2\left[T_H-x_2\right]\right)\right\} = \frac{1}{x_1A_T}\left(T_H-x_2\right)
\end{aligned}
$$

In order to obtain the linearized systems, we need to evaluate the partial derivatives at the equilibrium points. Thus, we obtain the following linearized systems:

$$
\begin{cases}
\begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 30 \end{bmatrix} + \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix}, \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} = \begin{bmatrix} 0.1329 \\ 0.0221 \end{bmatrix} + \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}, \begin{bmatrix} h(0) \\ T_T(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 30 \end{bmatrix} + \begin{bmatrix} \delta h(0) \\ \delta T_T(0) \end{bmatrix} \\
\begin{bmatrix} \delta \dot{h}(t) \\ \delta \dot{T}_T(t) \end{bmatrix} \approx \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.0517 \end{bmatrix} \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix} + \begin{bmatrix} 0.3333 & 0.3333 \\ -3.3333 & 20.0000 \end{bmatrix} \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}
\end{cases}
$$

$$
\begin{cases}
\begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 70 \end{bmatrix} + \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix}, \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} = \begin{bmatrix} 0.0443 \\ 0.1107 \end{bmatrix} + \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}, \begin{bmatrix} h(0) \\ T_T(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 70 \end{bmatrix} + \begin{bmatrix} \delta h(0) \\ \delta T_T(0) \end{bmatrix} \\
\begin{bmatrix} \delta \dot{h}(t) \\ \delta \dot{T}_T(t) \end{bmatrix} \approx \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.0517 \end{bmatrix} \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix} + \begin{bmatrix} 0.3333 & 0.3333 \\ -16.6667 & 6.6667 \end{bmatrix} \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}
\end{cases}
$$

$$\begin{cases} \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 30 \end{bmatrix} + \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix}, \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} = \begin{bmatrix} 0.2658 \\ 0.0443 \end{bmatrix} + \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}, \begin{bmatrix} h(0) \\ T_T(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 30 \end{bmatrix} + \begin{bmatrix} \delta h(0) \\ \delta T_T(0) \end{bmatrix} \\ \begin{bmatrix} \delta \dot{h}(t) \\ \delta \dot{T}_T(t) \end{bmatrix} \approx \begin{bmatrix} -0.0129 & 0 \\ 0 & -0.0258 \end{bmatrix} \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix} + \begin{bmatrix} 0.3333 & 0.3333 \\ -0.8333 & 5.0000 \end{bmatrix} \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix} \end{cases}$$

$$\begin{cases} \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 70 \end{bmatrix} + \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix}, \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix} = \begin{bmatrix} 0.0886 \\ 0.2215 \end{bmatrix} + \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix}, \begin{bmatrix} h(0) \\ T_T(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 70 \end{bmatrix} + \begin{bmatrix} \delta h(0) \\ \delta T_T(0) \end{bmatrix} \\ \begin{bmatrix} \delta \dot{h}(t) \\ \delta \dot{T}_T(t) \end{bmatrix} \approx \begin{bmatrix} -0.0129 & 0 \\ 0 & -0.0258 \end{bmatrix} \begin{bmatrix} \delta h(t) \\ \delta T_T(t) \end{bmatrix} + \begin{bmatrix} 0.3333 & 0.3333 \\ -4.1667 & 1.6667 \end{bmatrix} \begin{bmatrix} \delta q_C(t) \\ \delta q_H(t) \end{bmatrix} \end{cases}$$