2 Assignment

In the mandatory exercises for this assignment, you will practice with the computation of the state and output responses for LTI systems. An optional exercise will help you in gaining more insight into the relationship between the eigenvalues of the state matrix A and the behavior of an LTI system.

2.1 Computation of the state and output responses (mandatory)

Question 2.1

NOTE! You do not have to do any linearization in this exercise because the system is already linear! Consider the following system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.1}$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.2}$$

Determine the state and output responses $x(t)$ and $y(t)$ corresponding to an initial condition $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and an input signal $u(t) = 1(t)$, where $1(t)$ denotes the unit step:

$$1(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \tag{2.3}$$

Open the file `hw2a.slx` to compare the simulated output response of the state-space model with the calculated output response. You need to insert suitable values inside the gain and constant blocks, according to the solution that you calculated. *Note: Some calculations can be quite long and tedious. You can use some computational aid to help you in the process, for example WolframAlpha. Please explain in detail where and how you used computational aid.*

2.2 Computation of the state response for the pendulum

Question 2.2

We have seen that the equation of a pendulum is given by:

$$ml^2\ddot{\theta}(t) = f(t)l - mgl\sin\theta(t) \tag{2.4}$$

where m is the mass, l is the length of the rod, g denotes the gravity acceleration, $\theta(t)$ denotes the angle with respect to the vertical axis, and $f(t)$ is the force acting on the mass in the tangential direction. Let's consider $m = 1\,\text{kg}$, $l = 1\,\text{m}$ and $g = 10\,\text{m/s}^2$, and let's choose state variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$ with input variable $u(t) = f(t)$:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -10\sin x_1(t) + u(t) \end{cases} \tag{2.5}$$

We can linearize the pendulum around different operating point, and then analyze what happens if an initial condition different from the equilibrium is considered (state response). In this exercise, you should compute the state responses $x_d(t)$ corresponding to the initial deviation $\delta x(0) = \begin{bmatrix}\pi/6 & 0\end{bmatrix}^T$ for the equilibrium state corresponding to the constant input $\bar{u} = mg/2 = 5\,\text{N}$:

$$\bar{x}_d = \begin{bmatrix}\pi/6 \\ 0\end{bmatrix} \tag{2.6}$$

NOTE! In this exercise $\delta u(t) = 0$, since $u(t) = \bar{u}$. This means that the effect of the input is embedded into the equilibrium state $\bar{x}_d$, and you do not need to compute the convolution integral $\int_0^t e^{A(t-\tau)}Bd\tau$ (since it will be equal to 0).

2.3 State-space response (optional)

In this exercise, we are going to use the file *HW2statespace.exe*, which contains a ready-to-use simulator developed by UNED. The app is in Spanish, so you are referred to Fig. 2.1 for an English translation.

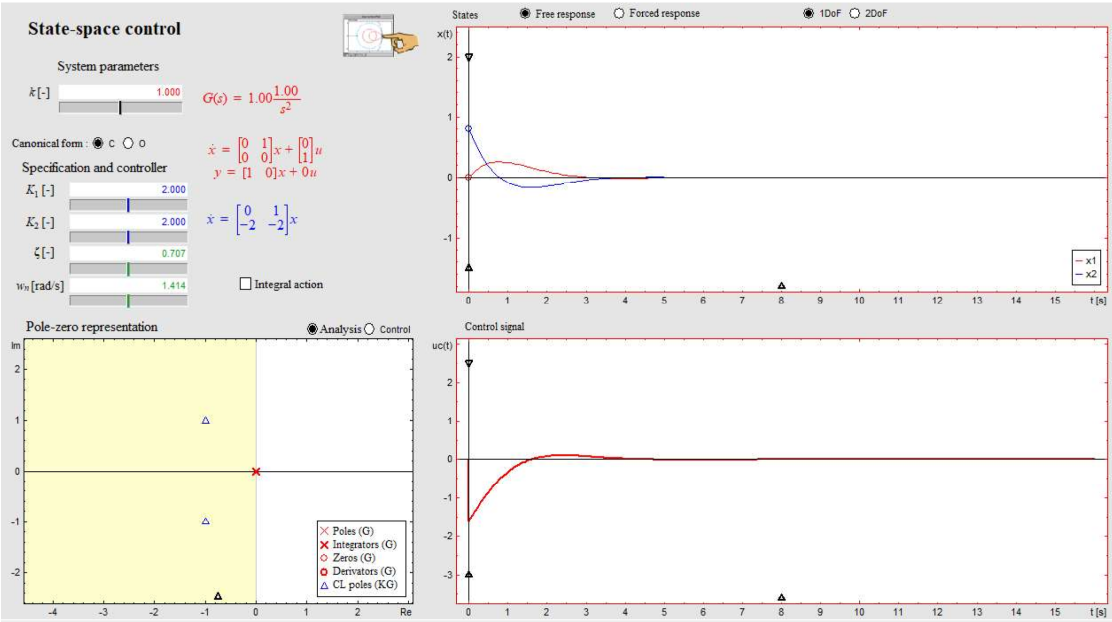

Figure 2.1: Software state-space.

We will use the simulator to get further insight about how the eigenvalues of a system are related to the time response. The simulated system is a second-order system with the following state equation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.7}$$

The input signal is chosen as[1]:

$$u(t) = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} x(t) \tag{2.8}$$

The interconnection of (2.7)-(2.8) leads to a system that behaves according to:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix} x(t) \tag{2.9}$$

Hence, it is clear that the eigenvalues of the state matrix in (2.9) would change according to the values of K_1 and K_2.

[1]This is called *state-feedback control law*, you could learn more about it in more advanced courses

Question 2.3

Change the values of K_1 and K_2 (negative values can be used) and see how the *closed-loop* eigenvalues of the state matrix in (2.9) (blue triangles in the pole-zero representation) are related to the states' free response in the right part of the software. Alternatively, you can switch from the *Analysis* to the *Control* mode so that you can move directly the eigenvalues on the pole-zero representation (green squares). What happens when the eigenvalues are moved towards the left? What happens when the imaginary part is increased? What happens when they are moved towards the right, possibly crossing the imaginary axis so that their real part becomes positive?