2 Assignment

In the mandatory exercises for this assignment, you will practice with the computation of the state and output responses for LTI systems. An optional exercise will help you in gaining more insight into the relationship between the eigenvalues of the state matrix A and the behavior of an LTI system.

2.1 Computation of the state and output responses (mandatory)

Question 2.1

NOTE! You do not have to do any linearization in this exercise because the system is already linear! Consider the following system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.1}$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.2}$$

Determine the state and output responses $x(t)$ and $y(t)$ corresponding to an initial condition $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and an input signal $u(t) = 1(t)$, where $1(t)$ denotes the unit step:

$$1(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \tag{2.3}$$

Open the file `hw2a.slx` to compare the simulated output response of the state-space model with the calculated output response. You need to insert suitable values inside the gain and constant blocks, according to the solution that you calculated. *Note: Some calculations can be quite long and tedious. You can use some computational aid to help you in the process, for example WolframAlpha. Please explain in detail where and how you used computational aid.*

Solution: The state response is given by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau$$

whereas the output response is given by:

$$y(t) = Cx(t)$$

A key element for computing $x(t)$ (and therefore $y(t)$) is the matrix exponential e^{At}. In order to compute it, we need to find the eigenvalues/eigenvectors of A. First, we calculate the characteristic polynomial $p(\lambda)$:

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -1 \\ -2 & \lambda+3 \end{vmatrix} = (\lambda+2)(\lambda+3) - (-1)(-2) = \lambda^2 + 5\lambda + 4$$

Then, we compute the eigenvalues as the roots of the characteristic polynomial:

$$p(\lambda) = 0 \Rightarrow \lambda^2 + 5\lambda + 4 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -4$$

To find the eigenvector u_1 associated to $\lambda_1 = -1$, we solve:

$$Au_1 = \lambda_1 u_1 \Rightarrow \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_1^a \\ u_1^b \end{bmatrix} = -1 \begin{bmatrix} u_1^a \\ u_1^b \end{bmatrix} \Rightarrow \begin{cases} -2u_1^a + u_1^b = -u_1^a \\ 2u_1^a - 3u_1^b = -u_1^b \end{cases}$$

By assigning arbitrarily $u_1^a = 1$, we obtain $u_1^b = 1$. Hence:

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find the eigenvector u_2 associated to $\lambda_2 = -4$, we solve:

$$Au_2 = \lambda_2 u_2 \Rightarrow \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_2^a \\ u_2^b \end{bmatrix} = -4 \begin{bmatrix} u_2^a \\ u_2^b \end{bmatrix} \Rightarrow \begin{cases} -2u_2^a + u_2^b = -4u_2^a \\ 2u_2^a - 3u_2^b = -4u_2^b \end{cases}$$

By assigning arbitrarily $u_2^a = 1$, then we get $u_2^b = -2$. Hence:

$$u_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Note that infinite choices are possible for the eigenvectors, for example:

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad u_1 = \begin{bmatrix} 1.73 \\ 1.73 \end{bmatrix} \quad u_1 = \begin{bmatrix} \pi \\ \pi \end{bmatrix} \quad u_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

are all eigenvectors associated to λ_1, whereas:

$$u_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1.73 \\ -3.46 \end{bmatrix} \quad u_2 = \begin{bmatrix} \pi \\ -2\pi \end{bmatrix} \quad u_2 = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

are all eigenvectors associated to λ_2.

Then, we can write down:

$$A = TDT^{-1} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

from which we can compute:

$$e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix}$$

The part of the state response corresponding to the initial condition $x(0)$ (*free response*) is given by:

$$e^{At}x(0) = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} \\ 2e^{-t} - 2e^{-4t} \end{bmatrix}$$

whereas the part corresponding to the input $u(t)$ is given by:

$$\begin{aligned} e^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau &= e^{At} \int_0^t \frac{1}{3} \begin{bmatrix} 2e^{\tau} + e^{4\tau} & e^{\tau} - e^{4\tau} \\ 2e^{\tau} - 2e^{4\tau} & e^{\tau} + 2e^{4\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 d\tau \\ &= e^{At} \frac{1}{3} \begin{bmatrix} \int_0^t \left(e^{\tau} - e^{4\tau}\right) d\tau \\ \int_0^t \left(e^{\tau} + 2e^{4\tau}\right) d\tau \end{bmatrix} = e^{At} \frac{1}{3} \begin{bmatrix} e^t - \frac{1}{4}e^{4t} - 1 + \frac{1}{4} \\ e^t + \frac{1}{2}e^{4t} - 1 - \frac{1}{2} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} e^t - \frac{1}{4}e^{4t} - \frac{3}{4} \\ e^t + \frac{1}{2}e^{4t} - \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{12}\left(e^{-4t} - 4e^{-t} + 3\right) \\ \frac{1}{6}\left(-e^{-4t} - 2e^{-t} + 3\right) \end{bmatrix} \end{aligned}$$

Consequently, the state response is given by:

$$x(t) = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{-4t} \\ 2e^{-t} - 2e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{1}{12}\left(e^{-4t} - 4e^{-t} + 3\right) \\ \frac{1}{6}\left(-e^{-4t} - 2e^{-t} + 3\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{12}\left(5e^{-4t} + 4e^{-t} + 3\right) \\ \frac{1}{6}\left(-5e^{-4t} + 2e^{-t} + 3\right) \end{bmatrix}$$

which means that the output response is given by:

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(t) = \frac{1}{12}\left(5e^{-4t} + 4e^{-t} + 3\right) + \frac{1}{3}\left(-5e^{-4t} + 2e^{-t} + 3\right) = \frac{1}{4}\left(-5e^{-4t} + 4e^{-t} + 5\right)$$

2.2 Computation of the state response for the pendulum

Question 2.2

We have seen that the equation of a pendulum is given by:

$$ml^2\ddot{\theta}(t) = f(t)l - mgl\sin\theta(t) \tag{2.4}$$

where m is the mass, l is the length of the rod, g denotes the gravity acceleration, $\theta(t)$ denotes the angle with respect to the vertical axis, and $f(t)$ is the force acting on the mass in the tangential direction. Let's consider $m = 1\,\text{kg}$, $l = 1\,\text{m}$ and $g = 10\,\text{m/s}^2$, and let's choose state variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$ with input variable $u(t) = f(t)$:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -10\sin x_1(t) + u(t) \end{cases} \tag{2.5}$$

We can linearize the pendulum around different operating point, and then analyze what happens if an initial condition different from the equilibrium is considered (state response). In this exercise, you should compute the state responses $x_d(t)$ corresponding to the initial deviation $\delta x(0) = \begin{bmatrix}\pi/6 & 0\end{bmatrix}^T$ for the equilibrium state corresponding to the constant input $\bar{u} = mg/2 = 5\,\text{N}$:

$$\bar{x}_d = \begin{bmatrix}\pi/6 \\ 0\end{bmatrix} \tag{2.6}$$

NOTE! In this exercise $\delta u(t) = 0$, since $u(t) = \bar{u}$. This means that the effect of the input is embedded into the equilibrium state $\bar{x}_d$, and you do not need to compute the convolution integral $\int_0^t e^{A(t-\tau)}Bd\tau$ (since it will be equal to 0).

Solution: The linearized system that describes the behavior of the state around $\bar{x}_d$ is:

$$\begin{cases} \begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} = \begin{bmatrix}\pi/6 \\ 0\end{bmatrix} + \begin{bmatrix}\delta x_1(t) \\ \delta x_2(t)\end{bmatrix}, u(t) = 5 + \delta u(t) \\ \begin{bmatrix}\delta\dot{x}_1(t) \\ \delta\dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -5\sqrt{3} & 0\end{bmatrix}\begin{bmatrix}\delta x_1(t) \\ \delta x_2(t)\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\delta u(t) \end{cases}$$

Then we can compute:

$$\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} = \begin{bmatrix}\pi/6 \\ 0\end{bmatrix} + e^{At}\begin{bmatrix}\delta x_1(0) \\ \delta x_2(0)\end{bmatrix}$$

To compute e^{At}, first we compute eigenvalues and eigenvectors. The eigenvalues are computed as the solution of the following second-order equation:

$$\det(\lambda I - A) = \begin{vmatrix}\lambda & -1 \\ 5\sqrt{3} & \lambda\end{vmatrix} = \lambda^2 + 5\sqrt{3} = 0$$

whose solutions are: $\lambda_1 = \sqrt{5\sqrt{3}} = \sqrt[4]{75}j$ and $\lambda_2 = \lambda_1^* = -\sqrt[4]{75}j$ Then, we can compute the eigenvector associated to λ_1:

$$Au_1 = \lambda_1 u_1 \Rightarrow \begin{bmatrix} 0 & 1 \\ -5\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} u_1^a \\ u_1^b \end{bmatrix} = \sqrt[4]{75}j \begin{bmatrix} u_1^a \\ u_1^b \end{bmatrix}$$

By assigning arbitrarily $u_1^a = 1$, we get $u_1^b = \sqrt[4]{75}j$, so a possible eigenvector u_1 is:

$$u_1 = \begin{bmatrix} 1 \\ \sqrt[4]{75}j \end{bmatrix}$$

The eigenvector associated to λ_2 is the complex conjugate vector (but we will not need it):

$$u_2 = u_1^* \begin{bmatrix} 1 \\ -\sqrt[4]{75}j \end{bmatrix}$$

Then, we can write down the following:

$$\begin{aligned} A = T\Sigma T^{-1} &= \begin{bmatrix} \mathrm{Re}(u_1) & \mathrm{Im}(u_1) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\lambda_1) & \mathrm{Im}(\lambda_1) \\ -\mathrm{Im}(\lambda_1) & \mathrm{Re}(\lambda_1) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(u_1) & \mathrm{Im}(u_1) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{75} \end{bmatrix} \begin{bmatrix} 0 & \sqrt[4]{75} \\ -\sqrt[4]{75} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{75} \end{bmatrix}^{-1} \end{aligned}$$

from which we obtain:

$$e^{At} = Te^{\Sigma t}T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{75} \end{bmatrix} \begin{bmatrix} \cos\left(\sqrt[4]{75}t\right) & \sin\left(\sqrt[4]{75}t\right) \\ -\sin\left(\sqrt[4]{75}t\right) & \cos\left(\sqrt[4]{75}t\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt[4]{75} \end{bmatrix} = \begin{bmatrix} \cos\left(\sqrt[4]{75}t\right) & \frac{\sin\left(\sqrt[4]{75}t\right)}{\sqrt[4]{75}} \\ -\sqrt[4]{75}\sin\left(\sqrt[4]{75}t\right) & \cos\left(\sqrt[4]{75}t\right) \end{bmatrix}$$

Then:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\left(\sqrt[4]{75}t\right) & \frac{\sin\left(\sqrt[4]{75}t\right)}{\sqrt[4]{75}} \\ -\sqrt[4]{75}\sin\left(\sqrt[4]{75}t\right) & \cos\left(\sqrt[4]{75}t\right) \end{bmatrix} \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi/6 + \pi/6\cos\left(\sqrt[4]{75}t\right) \\ -\pi/6\sqrt[4]{75}\sin\left(\sqrt[4]{75}t\right) \end{bmatrix}$$