3 Assignment

In this assignment, which comprises only mandatory exercises, you will practice with using the Laplace transform and the transfer function in order to compute the output response of an LTI system.

You will find the following Laplace transforms useful:

$$\mathcal{L}\{e^{at}1(t)\} = \frac{1}{s-a} \qquad \mathcal{L}\{\sin(\omega t)1(t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{\cos(\omega t)1(t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \qquad \mathcal{L}\{te^{at}1(t)\} = \frac{1}{(s-a)^2} \qquad \mathcal{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$$

3.1 Laplace transform (mandatory)

Question 3.1

Given the following differential equation:

$$\dot{y}(t) = -3y(t) + u(t) \tag{3.1}$$

with initial value $y(0) = 5$. Assume that the input variable $u(t)$ is a step signal of amplitude 2 at time $t = 0$.

1. Compute the corresponding output response $y(t)$ using the Laplace transform.
2. Calculate the steady-state value of $y(t)$ (the final value of $y(t)$ when $t \to \infty$) using the Final Value Theorem. Then, calculate the steady-state value y_{ss} using the $y(t)$ computed at point 1. Are the two computed values the same?
3. According to the time-derivative property of the Laplace transform, $\mathcal{L}\{\dot{y}(t)\} = sY(s) - y_0$. Under the assumption that $y_0 = 0$, we obtain the following from (3.1):

$$sY(s) = -3Y(s) + U(s) \qquad \Rightarrow \qquad Y(s) = H(s)U(s) = \frac{1}{s+3}U(s) \tag{3.2}$$

   where $H(s)$ is the *transfer function*. Using MATLAB functions `tf` and `step`, write a code that plots the step response of (3.2). Then, using the Simulink blocks `Step`, `Transfer Fcn` and `Scope`, perform a simulation of the step response. Are the obtained signals $y(t)$ the same in both cases? Compare them with the expression of $y(t)$ that you obtained at point 1 of this exercise, and discuss similarities/differences.

Solution: 1. The Laplace transform of (3.1) is given by:

$$sY(s) - y(0) = -3Y(s) + U(s)$$

Since $u(t)$ is a step signal of amplitude 2, we have:

$$U(s) = \frac{2}{s}$$

We can compute:

$$(s+3)Y(s) = y(0) + U(s) \quad \Rightarrow \quad (s+3)Y(s) = 5 + \frac{2}{s} = \frac{5s+2}{s}$$

which leads to:

$$Y(s) = \frac{5s+2}{s(s+3)}$$

We can perform the partial fraction decomposition:

$$Y(s) = \frac{r_1}{s} + \frac{r_2}{s+3}$$

with:

$$r_1 = [Y(s)s]_{s=0} = \left[\frac{5s+2}{s+3}\right]_{s=0} = \frac{2}{3}$$
$$r_2 = [Y(s)(s+3)]_{s=-3} = \left[\frac{5s+2}{s}\right]_{s=-3} = \frac{13}{3}$$

So, we obtain:

$$Y(s) = \frac{2/3}{s} + \frac{13/3}{s+3} \quad \Rightarrow \quad y(t) = \left(\frac{2}{3} + \frac{13}{3}e^{-3t}\right) 1(t)$$

2. The final value theorem states that:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{5s+2}{s+3} = \frac{2}{3} = y_{ss}$$

On the other hand, taking into account the previously computed solution:

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\frac{2}{3} + \frac{13}{3}e^{-3t}\right) 1(t) = \frac{2}{3}$$

so the steady-state values calculated in these two different ways are the same.

3. The MATLAB code that plots the step response of (3.2) is the following:

```
mytf = tf(1,[1 3]);
step(mytf)
```

which returns the plot in Fig. 3.1. The Simulink implementation is shown in Fig. 3.2, and the results are illustrated in Fig. 3.3. The plots are similar, the only relevant differences are the lengths of the simulations and the time at which the step induces the response (in the Simulink case, the step response starts at time 1 s - you can open the Simulink block `Step` to understand why this happens). The comparison with the solution of point 1 $y(t) = \left(\frac{2}{3} + \frac{13}{3}e^{-3t}\right) 1(t)$, implemented using the following code:

```
T = 0:0.01:10;
Y = 2/3+13/3*exp(-3*T);
plot(T,Y); xlabel('Time (seconds)'); ylabel('Amplitude'); title('Response y(t)')
```

whose result is given in Fig. 3.4 shows a very different result. This is due to two facts: 1) the initial condition, which is different from 0; 2) the step signal introduced as input at point 1 of this exercise has amplitude equal to 2. It can be seen that after a certain time, the part of the response due to the initial condition vanishes (this is due to *stability*, we will discuss this concept later in the course), and the response converges to a value which is exactly the double of the steady-state value in Figs. 3.1-3.3 (this is due to linearity).

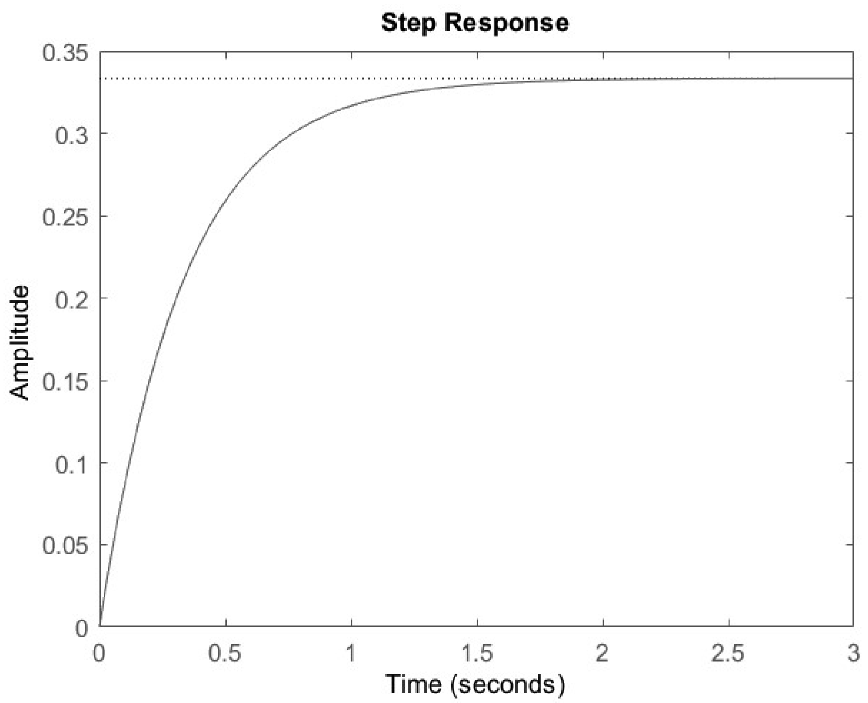

Figure 3.1: Step response of (3.2) (MATLAB).

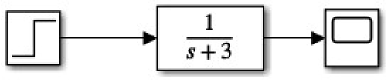

Figure 3.2: Implementation of (3.2) (SIMULINK).

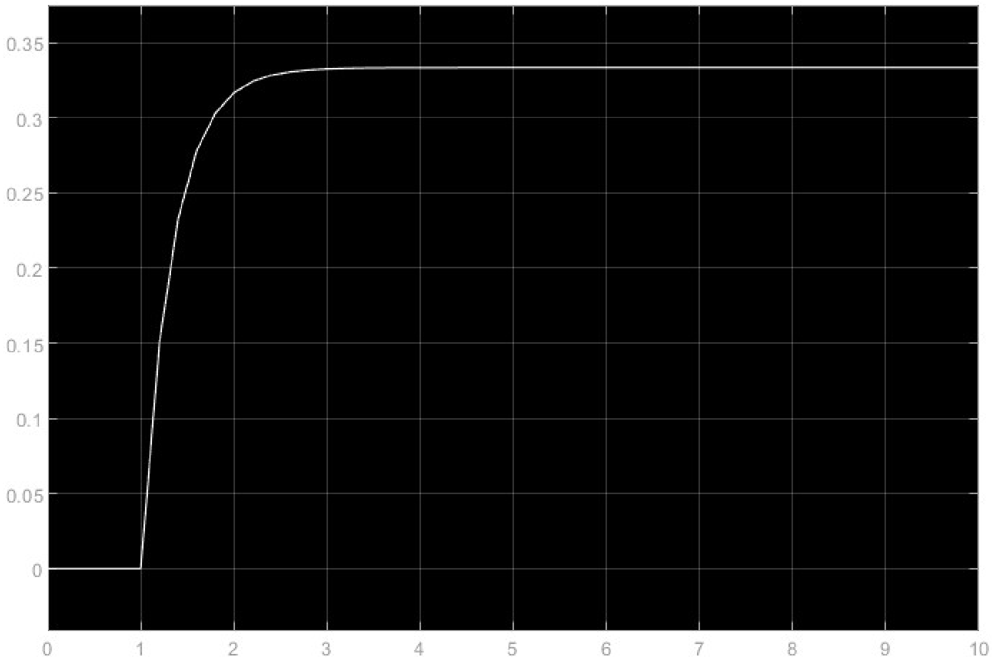

Figure 3.3: Step response of (3.2) (SIMULINK).

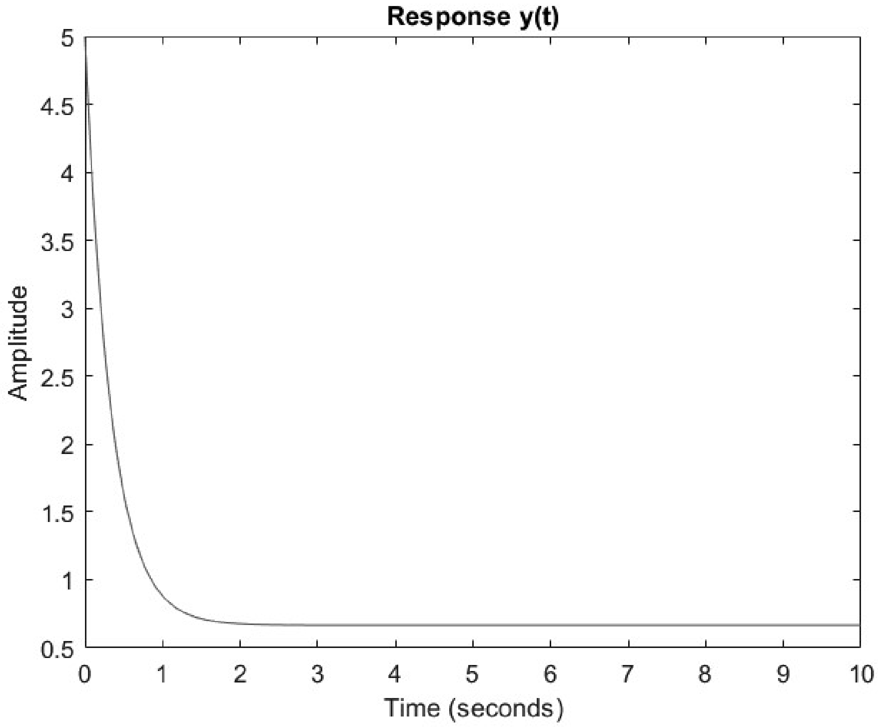

Figure 3.4: Response $y(t) = \left(\frac{2}{3} + \frac{13}{3}e^{-3t}\right) 1(t)$.

3.2 Transfer function (mandatory)

Question 3.2

Consider the mass-spring-damper system in Fig. 3.5, where y denotes the position, F is the applied force, D is the damping coefficient, K is the spring constant. By assuming that the damping force F_d is proportional to the velocity, and that the spring force F_s is proportional to the position of the mass, and such that $F_s = 0$ when $y = 0$, then the following equation is obtained from the force balance:

$$m\ddot{y}(t) = F(t) - D\dot{y}(t) - Ky(t) \tag{3.3}$$

Calculate the transfer function from the force F to the position y.

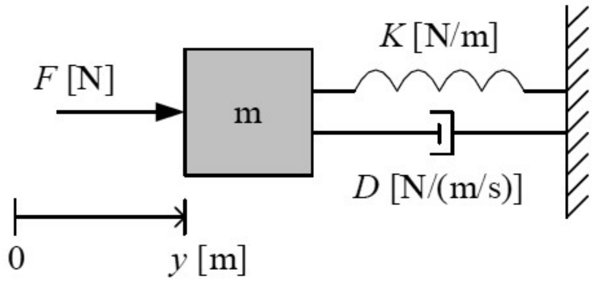

Figure 3.5: Mass-spring-damper system.

Solution: The transfer function from $F(s)$ to $Y(s)$ is given by:

$$H(s) = \frac{1}{ms^2 + Ds + K}$$

3.3 Output response using the transfer function (mandatory)

Question 3.3

1. Consider an LTI system described by the transfer function $H(s) = \frac{1}{(s+2)}$. Compute the output response $y_1(t)$ which corresponds to an input signal $u_1(t) = \sin(\omega t)1(t)$.
2. Imagine that the transfer function had been $H(s) = \frac{1}{(s+4)}$ instead. Which term in $y_1(t)$ could you change instantaneously without performing any calculation, and how? *Note: Of course, in order to get the correct entire expression for $y_1(t)$ you would still need to perform again all the calculations, you can do this as an optional exercise for familiarizing yourself further with the involved calculations.*
3. Consider an LTI system described by the transfer function $H(s) = \frac{s-1}{(s+1)^2}$. Compute

the output response $y_2(t)$ which corresponds to the input signal $u_2(t) = 1(t)$.

Solution: 1. We have:

$$Y_1(s) = H(s)U_1(s) = \frac{\omega}{(s+2)(s^2+\omega^2)}$$

We can perform the partial fraction expansion of the above expression as follows:

$$Y_1(s) = \frac{r_1}{s+2} + \frac{r_2}{s-j\omega} + \frac{r_2^*}{s+j\omega}$$

where the residues r_1, r_2, r_2^* can be computed as:

$$r_1 = [Y_1(s)(s+2)]_{s=-2} = \left[\frac{\omega}{(s^2+\omega^2)}\right]_{s=-2} = \frac{\omega}{\omega^2+4}$$

$$r_2 = [Y_1(s)(s-j\omega)]_{s=j\omega} = \left[\frac{\omega}{(s+2)(s+j\omega)}\right]_{s=j\omega} = \frac{\omega}{(2+j\omega)2j\omega}$$

$$r_2^* = [Y(s)(s+j\omega)]_{s=-j\omega} = \left[\frac{\omega}{(s+2)(s-j\omega)}\right]_{s=-j\omega} = \frac{\omega}{(2-j\omega)(-2j\omega)}$$

Then:

$$\mathcal{L}^{-1}\left\{\frac{r_1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{\omega}{\omega^2+4}\frac{1}{s+2}\right\} = \frac{\omega}{\omega^2+4}e^{-2t}1(t)$$

In order to compute the inverse Laplace transform of the remaining term, it is useful to rewrite r_2 in the form $\rho e^{j\varphi}$. We can write:

$$r_2 = \frac{\omega(2-j\omega)(-2j\omega)}{(2+j\omega)(2-j\omega)2j\omega(-2j\omega)} = -\frac{\omega^2(\omega+2j)}{4\omega^2(\omega^2+4)} = -\frac{1}{2}\frac{\omega+2j}{\omega^2+4}$$

so:

$$\rho = |r_2| = \frac{1}{2}\frac{\sqrt{\omega^2+4}}{\omega^2+4}$$

$$\varphi = \arg(r_2) = -\pi + \arctan\frac{2}{\omega}$$

from which we obtain:

$$\mathcal{L}^{-1}\left\{\frac{r_2}{s-j\omega} + \frac{r_2^*}{s+j\omega}\right\} = 2\rho\cos(\omega t+\varphi)1(t) = \frac{\sqrt{\omega^2+4}}{\omega^2+4}\cos(\omega t - \pi + \arctan\frac{2}{\omega})1(t)$$

Then, we can conclude that:

$$y_1(t) = \left(\frac{\omega}{\omega^2+4}e^{-2t} + \frac{\sqrt{\omega^2+4}}{\omega^2+4}\cos\left(\omega t - \pi + \arctan\frac{2}{\omega}\right)\right)1(t)$$

2. The term e^{-2t} would change into e^{-4t} due to the pole of the transfer function $H(s)$ being changed from $s = -2$ to $s = -4$.

3. In this case, we have:

$$Y_2(s) = H(s)U_2(s) = \frac{s-1}{s(s+1)^2}$$

which can be rewritten using the partial fraction expansion as follows:

$$Y_2(s) = \frac{r_1}{s} + \frac{r_{21}}{(s+1)^2} + \frac{r_{22}}{s+1}$$

where the residues r_1, r_{21}, r_{22} can be computed as:

$$r_1 = [Y_2(s)s]_{s=0} = \left[\frac{(s-1)}{(s+1)^2}\right]_{s=0} = -1$$

$$r_{21} = \left[Y_2(s)(s+1)^2\right]_{s=-1} = \left[\frac{s-1}{s}\right]_{s=-1} = 2$$

$$r_{22} = \left[\frac{d}{ds}\left\{Y_2(s)(s+1)^2\right\}\right]_{s=-1} = \left[\frac{d}{ds}\left\{\frac{s-1}{s}\right\}\right]_{s=-1} = \left[\frac{1}{s^2}\right]_{s=-1} = 1$$

Then we get:

$$y_2(t) = \left(-1 + 2te^{-t} + e^{-t}\right) 1(t)$$