4 Assignment

In the mandatory exercises of this assignment, you will practice with the relationship between the state-space and the transfer function. You will also get a deeper understanding of BIBO stability and internal stability. Finally, you will practice with the simplification of a block diagram.

4.1 From state-space to transfer functions (mandatory)

Question 4.1

Obtain the transfer function of the system defined by:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + x_2(t) \\ \dot{x}_2(t) = -x_2(t) + x_3(t) \\ \dot{x}_3(t) = -2x_3(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{4.1}$$

Discuss about the external (BIBO) stability and internal stability of this system.

Solution: The system (4.1) can be put into a compact matrix form by defining the following matrices:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad D = 0$$

Hence, we can calculate the transfer function $H(s)$ as follows:

$$H(s) = C(sI - A)^{-1}B + D$$

The matrix $(sI - A)^{-1}$ is given by[1]:

$$(sI - A)^{-1} = \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$

[1]In WolframAlpha: *inv*$(\{\{s+1, -1, 0\}, \{0, s+1, -1\}, \{0, 0, s+2\}\})$.

so that:

$$H(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{(s+1)^2(s+2)}$$

The external stability (BIBO) can be assessed by looking at the poles of the transfer function. The computed $H(s)$ has a double pole in -1 and another pole in -2. All the poles are in the left half of the complex plane $\mathrm{Re}(s) < 0$, so the system is BIBO stable. The internal stability is assessed by looking at the eigenvalues of A, which are the solutions of $\det(sI - A) = 0$:

$$\det(sI - A) = (s+1)^2(s+2) = 0$$

which shows that A has a double eigenvalue in -1 and an eigenvalue in -2 (note that they correspond to the poles of $H(s)$: there is no zero-pole cancellation). Since all the eigenvalues are in the left half of the complex plane $\mathrm{Re}(\lambda) < 0$, the system is asymptotically stable.

Question 4.2

Obtain the transfer function of the system defined by:

$$\begin{cases} \dot{x}_1(t) = -39x_1(t) + 84x_2(t) + 7u(t) \\ \dot{x}_2(t) = -18x_1(t) + 39x_2(t) + 3u(t) \\ y(t) = x_1(t) - 2x_2(t) \end{cases} \tag{4.2}$$

Discuss about the external (BIBO) stability and internal stability of this system.

Solution: The system (4.2) can be put into a compact matrix representation with:

$$A = \begin{bmatrix} -39 & 84 \\ -18 & 39 \end{bmatrix} \qquad B = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -2 \end{bmatrix} \qquad D = 0$$

so that:

$$(sI - A)^{-1} = \begin{bmatrix} s+39 & -84 \\ 18 & s-39 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s-39}{s^2-9} & \frac{84}{s^2-9} \\ \frac{-18}{s^2-9} & \frac{s+39}{s^2-9} \end{bmatrix}$$

and:

$$H(s) = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{s-39}{s^2-9} & \frac{84}{s^2-9} \\ \frac{-18}{s^2-9} & \frac{s+39}{s^2-9} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \frac{s-3}{s^2-9} = \frac{1}{s+3}$$

The transfer function $H(s)$ has a single pole in -3, which means that it is BIBO stable. However, the eigenvalues of the matrix A are -3 and 3 (the pole in 3 was canceled by the zero in 3) which means that the system is unstable when the behavior under perturbations of the initial state is considered.

4.2 Asymptotic and BIBO stability (mandatory)

Question 4.3

Consider the system given by the following equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - x_2(t)^2 x_1(t) \\ \dot{x}_2(t) = -5x_1(t) - 2x_2(t) \end{cases} \tag{4.3}$$

(a) Linearize the system about the equilibrium point $(\bar{x}_1, \bar{x}_2) = (0, 0)$ and discuss the internal stability of the linearized system.

(b) Consider a system described by the differential equation $\ddot{x}(t) + p\dot{x}(t) + qx(t) = u(t)$. For which values of the parameters p and q is the system BIBO stable?

(c) Plot the impulse response of the system for $p = 2$ and $q = 5$ (you can use MATLAB functions `tf` and `impulse` to do this).

(d) Now let $p = 2$ and $q = -5$. Is the system BIBO stable? Plot the impulse response.

Solution: (a) The linearized system is given by:

$$\begin{cases} x(t) = \delta x(t), x(0) = \delta x(0) \\ \delta\dot{x}(t) \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\substack{x_1 = \bar{x}_1 \\ x_2 = \bar{x}_2}} \delta x(t) = \begin{bmatrix} -x_2^2 & 1 - 2x_1x_2 \\ -5 & -2 \end{bmatrix}_{\substack{x_1 = 0 \\ x_2 = 0}} \delta x(t) = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \delta x(t) \end{cases}$$

The eigenvalues of the state matrix are computed as:

$$\det(sI - A) = \det\left(\begin{bmatrix} s & -1 \\ 5 & s+2 \end{bmatrix}\right) = s^2 + 2s + 5 = 0$$

The matrix A has a pair of complex conjugate eigenvalues in $-1 \pm 2j$, which are in the left half of the complex plane, so the linearized system is asymptotically stable (as a consequence, the origin of the state-space is an asymptotically stable equilibrium point).

(b) The transfer function corresponding to the differential equation is:

$$H(s) = \frac{1}{s^2 + ps + q}$$

The poles of $H(s)$ are in:

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

The system is BIBO stable as long as the real parts of the poles is strictly negative. If $p \leq 0$, this condition is not satisfied. It $p > 0$, then what must happen is that:

$$-p + \sqrt{p^2 - 4q} < 0 \quad \Rightarrow \quad \sqrt{p^2 - 4q} < p \quad \Rightarrow \quad p^2 - 4q < p^2 \quad \Rightarrow \quad q > 0$$

Hence, the system is BIBO stable as long as $p > 0$ and $q > 0$.

(c) The impulse response can be plotted with the command `impulse(tf(1,[1 2 5]))` (see Fig. 4.1).

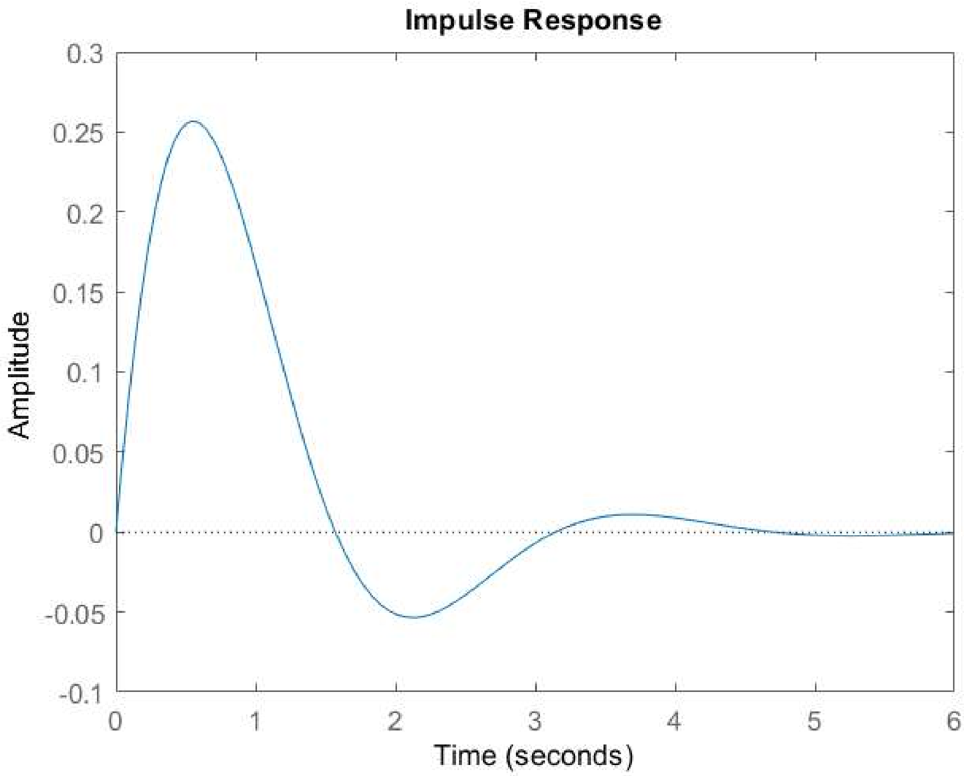

Figure 4.1: Impulse response with $p = 2$ and $q = 5$.

(d) When $p = 2$ and $q = -5$, the system is not BIBO stable. This is clearly shown by the impulse response plotted with `impulse(tf(1,[1 2 -5]))` and shown in Fig. 4.2.

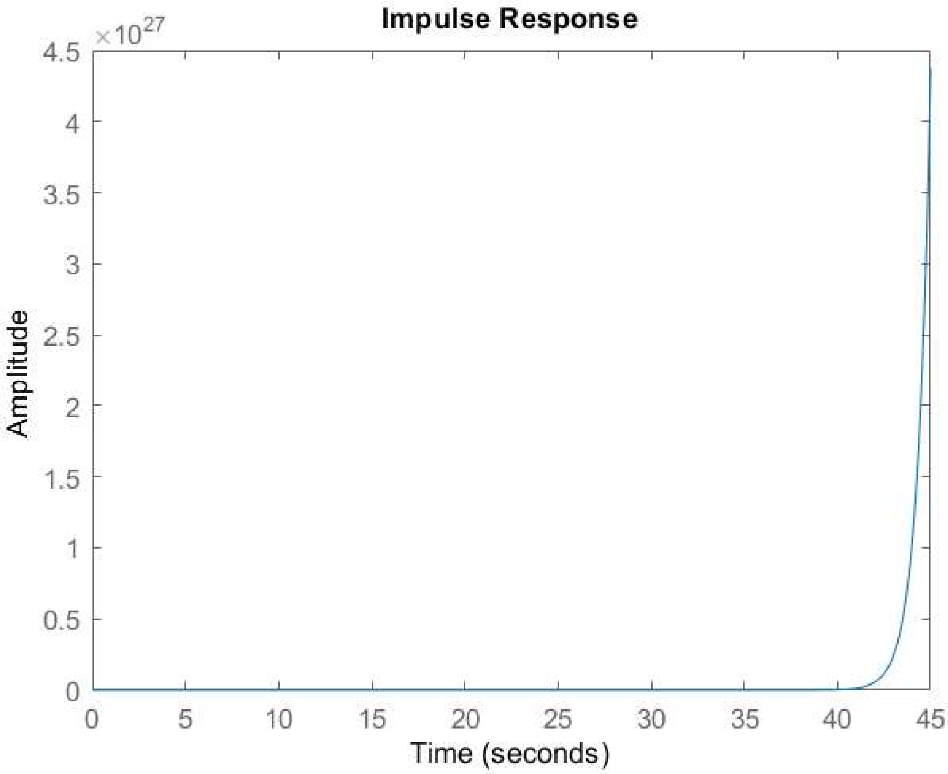

Figure 4.2: Impulse response with $p = 2$ and $q = -5$.

4.3 Block diagrams (mandatory)

Question 4.4

Simplify the block diagram shown in the figure, so that you obtain the equivalent transfer function $H_{RY}(s)$ from $R(s)$ to $Y(s)$. Using Simulink, show that the responses obtained using the original block diagram and the simplified block diagrams are equivalent (*you can use any type of input signal to show the equivalence*). What are the static gain $H_{RY}(0)$, the zeros and the poles of $H_{RY}(s)$? What type of exponential/trigonometric functions would we find in the step response of $H_{RY}(s)$? Is $H_{RY}(s)$ a BIBO stable system? What can we say about the internal stability of the underlying state-space representation?

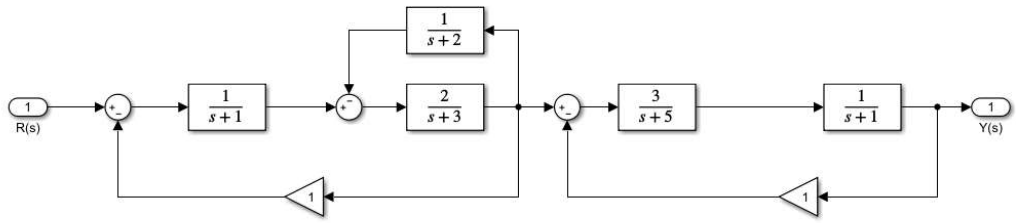

Figure 4.3: Block diagram of the system.

Solution: The first step to solve this exercise is to identify interconnections among series, parallel and negative-feedback. As shown in Fig. 4.4, there are a negative-feedback interconnection (cyan color) and a series interconnection (green color).

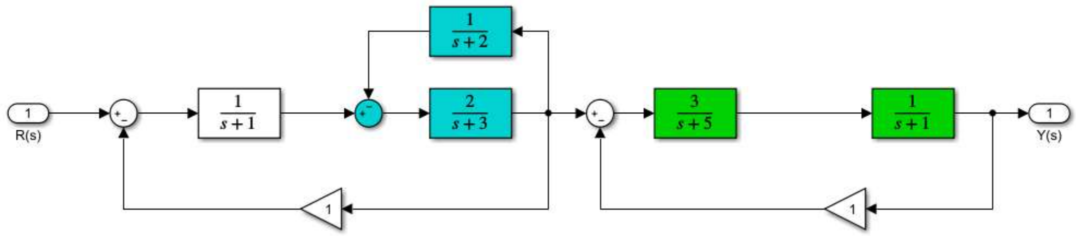

Figure 4.4: Block diagram of the system (negative-feedback and series interconnections).

By recalling that the equivalent transfer function of a feedback interconnection $H_{nf}(s)$ is given by:

$$H_{nf}(s) = \frac{H_{dl}(s)}{1 + H_{dl}(s)H_{fl}(s)}$$

where $H_{dl}(s)$ is the transfer function on the direct loop and $H_{fl}(s)$ is the transfer function on the feedback loop, whereas the equivalent transfer function of a feedback interconnection $H_{se}(s)$ is given by:

$$H_{se}(s) = H_2(s)H_1(s)$$

where $H_1(s)$ is the first transfer function and $H_2(s)$ is the second transfer function in the series interconnection, then we can calculate:

$$H_{cyan}(s) = \frac{\frac{2}{s+3}}{1 + \frac{2}{s+3}\frac{1}{s+2}} = \frac{\frac{2}{s+3}}{\frac{(s+3)(s+2)+2}{(s+3)(s+2)}} = \frac{2s+4}{s^2+5s+8}$$

$$H_{green}(s) = \frac{1}{s+1}\frac{3}{s+5} = \frac{3}{s^2+6s+5}$$

So, we can simplify the block diagram as shown in Fig. 4.5, in which we can identify another series interconnection (yellow color) and a negative-feedback interconnection (orange color).

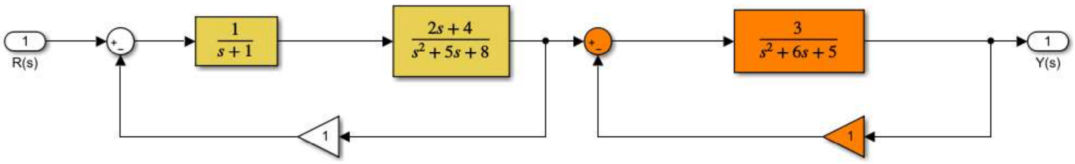

Figure 4.5: Block diagram of the system (series and negative-feedback interconnections).

By calculating the equivalent transfer functions, we get:

$$H_{yellow}(s) = \frac{2s+4}{s^2+5s+8}\frac{1}{s+1} = \frac{2s+4}{s^3+6s^2+13s+8}$$

$$H_{orange}(s) = \frac{\frac{3}{s^2+6s+5}}{1+\frac{3}{s^2+6s+5}} = \frac{3}{s^2+6s+8}$$

The obtained block diagram can be simplified further by identifying another feedback interconnection, which corresponds to:

$$H_{red}(s) = \frac{\frac{2s+4}{s^3+6s^2+13s+8}}{1+\frac{2s+4}{s^3+6s^2+13s+8}} = \frac{2s+4}{s^3+6s^2+15s+12}$$

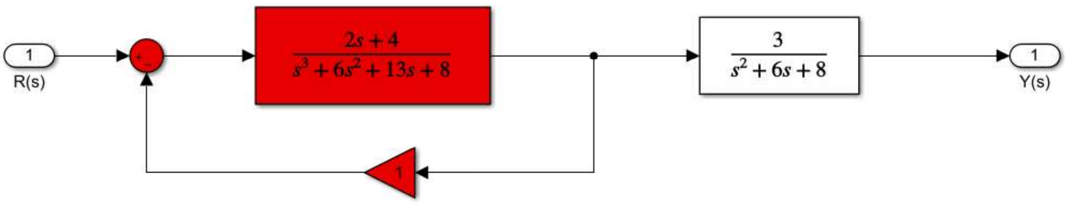

Figure 4.6: Block diagram of the system (negative-feedback interconnection).

Finally, by considering the series interconnection of $H_{red}(s)$ and $H_{orange}(s)$, we get the solution:

$$H_{RY}(s) = H_{orange}(s)H_{red}(s) = \frac{6s + 12}{s^5 + 12s^4 + 59s^3 + 150s^2 + 192s + 96}$$

The static gain is:

$$H_{RY}(0) = 12/96 = 1/8$$

The overall system has a single zero in -2, and five poles in: -4, $-2.298+1.8073j$, $-2.298-1.8073j$, -2 and -1.4039. Hence, we will find the following terms in the step response:

$$1(t), e^{-4t}1(t), e^{-2.298t}\cos(1.8073t + \phi)1(t), e^{-2t}1(t), e^{-1.4039t}1(t)$$

Since all the poles of $H_{RY}(s)$ have a negative real part, the system $H_{RY}(s)$ is BIBO stable. We cannot say anything about the internal stability of the underlying state-space representation, because we do not know if any of the transfer function in the scheme there was a zero-pole cancellation. In case this did not happen or if the cancellations involved zero-pole pairs with negative real part, the overall system would be asymptotically stable as well. Otherwise, it might be only marginally stable or even unstable.