5 Assignment

In this assignment, you are first going to work with two mandatory assignments that will help you in understanding the theoretical behavior of first-order and second-order systems. Afterwards, another mandatory exercise will make you work with a simulated real-world scenario concerning a first-order system available in our kitchens (a hob, på norks kokeplate). Finally, you are advised to take a look at the three optional exercises at the end of this document, which would strengthen your understanding of the behavior of first-order, second-order (without zeros) and second-order (with a real zero) systems.

5.1 First-order systems (mandatory)

Question 5.1

Given the following transfer functions:

$$H_1(s) = \frac{1}{4s+1} \tag{5.1}$$

$$H_2(s) = \frac{2}{4s+5} \tag{5.2}$$

(a) Find the poles and the 2% and 5% settling times for these transfer functions.

(b) Let the input signal $u(t)$ be a unit step. Find the corresponding step responses $y_1(t)$ and $y_2(t)$. *Hint: Write $H_1(s)$ and $H_2(s)$ in the standard form $H(s) = H(0)/(1 + s\tau)$.*

(c) Find the steady-state value of $y_1(t)$ and $y_2(t)$.

(d) Draw the step responses $y_1(t)$ and $y_2(t)$.

Solution: (a) A pole is a value of s that makes the transfer function singular. To find the pole(s), we set the denominator of the transfer function to zero, and solve the equation w.r.t. s. For $H_1(s)$, we obtain:

$$4s + 1 = 0 \quad \Rightarrow \quad s = -1/4$$

while for $H_2(s)$, we obtain:

$$4s + 5 = 0 \quad \Rightarrow \quad s = -5/4$$

In order to find the 2% and 5% settling times, we need to determine the time constant τ, which means rewriting the transfer function in the form $H(0)/(1 + s\tau)$. $H_1(s)$ is already in this form, so $\tau_1 = 4$ s. On the other hand:

$$H_2(s) = \frac{2}{4s+5} = \frac{2/5}{1+s4/5}$$

which shows that $\tau_2 = 4/5\,\text{s} = 0.8\,\text{s}$ (*be careful: this s here means seconds, whereas the s in the transfer function is the complex frequency of the Laplace domain!!*). Hence:

$$H_1(s) : T_{s5\%} = 3\tau_1 = 12\,\text{s} \quad T_{s2\%} = 4\tau_1 = 16\,\text{s} H_2(s) : T_{s5\%} = 3\tau_2 = 2.4\,\text{s} \quad T_{s2\%} = 4\tau_2 = 3.2\,\text{s}$$

(b) The input signal is $u(t) = 1(t)$, which corresponds to:

$$U(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}$$

Then, the Laplace transform of the output is given by:

$$Y(s) = \frac{H(0)}{1+s\tau} \cdot \frac{1}{s} = H(0)\left[\frac{1}{s} - \frac{1}{s+1/\tau}\right]$$

which is inverse-Laplace-transformed into:

$$y(t) = H(0)\left(1 - e^{-t/\tau}\right)1(t)$$

So we have:

$$y_1(t) = \left(1 - e^{-t/4}\right)1(t)$$
$$y_2(t) = \frac{2}{5}\left(1 - e^{-\frac{5}{4}t}\right)1(t)$$

(c) The steady-state values of $y_1(t)$ and $y_2(t)$ can be found by allowing $t \longrightarrow +\infty$ in the corresponding expressions:

$$y_{1ss} = \lim_{t\to+\infty} y_1(t) = \lim_{t\to+\infty}\left(1 - e^{-t/4}\right)1(t) = 1$$
$$y_{2ss} = \lim_{t\to+\infty} y_2(t) = \lim_{t\to+\infty}\frac{2}{5}\left(1 - e^{-\frac{5}{4}t}\right)1(t) = \frac{2}{5}$$

The same result can be found by applying the final value theorem to the functions $Y_1(s)$ and $Y_2(s)$.

(d) The step responses $y_1(t)$ and $y_2(t)$ are shown in Fig. 5.1. They can be generated in MATLAB by writing:

```
>> H1 = tf(1,[4 1]);
>> H2 = tf(2,[4 5]);
>> step(H1,H2)
```

It can be observed that:

- $y_1(t)$ and $y_2(t)$ reach csteady-state values corresponding to $H_1(0) = 1$ and $H_2(0) = 2/5 = 0.4$, respectively;
- $y_1(t)$ and $y_2(t)$ reach 63% of their steady-state values after $\tau_1 = 4$ s and $\tau_2 = 0.8$ s, respectively;
- $y_1(t)$ and $y_2(t)$ reach 95% of their steady-state values after $3\tau_1 = 12$ s and $3\tau_2 = 2.4$ s, respectively;
- $y_1(t)$ and $y_2(t)$ reach 98% of their steady-state values after $4\tau_1 = 16$ s and $4\tau_2 = 3.2$ s, respectively.

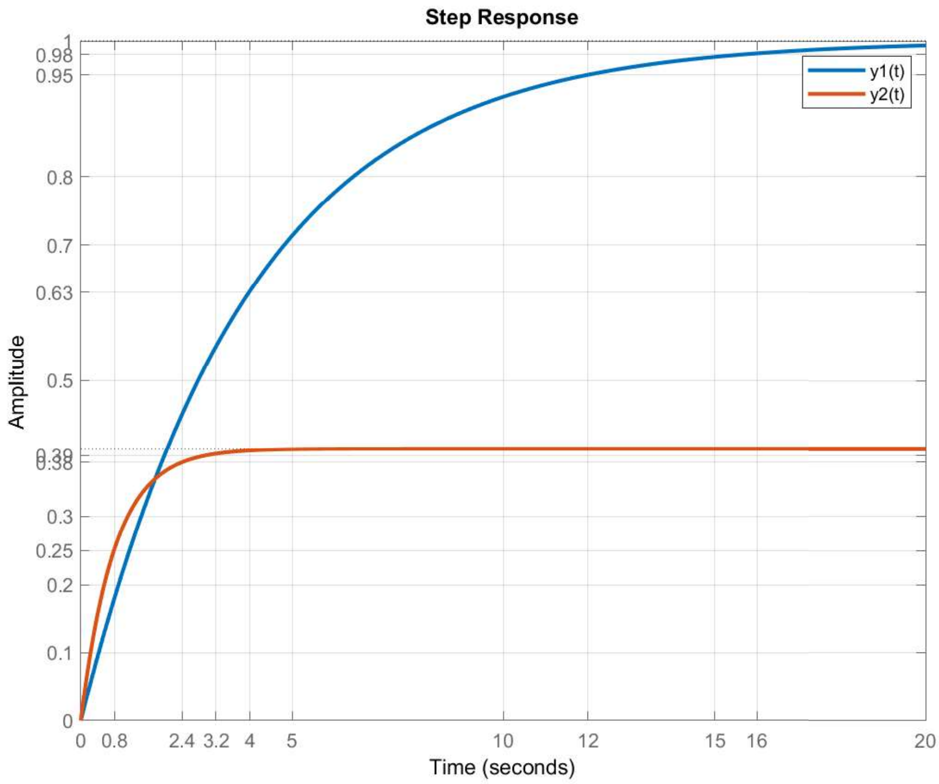

Figure 5.1: Step responses $y_1(t)$ and $y_2(t)$.

5.2 Second-order systems (mandatory)

Question 5.2

For each of the following transfer functions:

$$H_1(s) = \frac{15}{s^2 + 1.6s + 16} \tag{5.3}$$

$$H_2(s) = \frac{5}{2s^2 + 8s + 8} \tag{5.4}$$

$$H_3(s) = \frac{10}{s^2 + 7s + 10} \tag{5.5}$$

(a) Find the poles of the transfer function.

(b) Find the system's static gain $H(0)$, the natural frequency ω_0 and the damping ratio ζ. Determine the type of system (*undamped, underdamped, critically damped* or *overdamped*).

(c) For the underdamped systems from point (b), find the maximum overshoot, the 5% settling time, the 2% settling time, the peak time and the rise time.

(d) Using the final value theorem, determine the steady-state value when $u(t) = 1(t)$.

(e) Using MATLAB, obtain the responses of $y_1(t)$, $y_2(t)$ and $y_3(t)$ to the unit step $u(t) = 1(t)$, and check that they correspond to the values obtained at point (d). Moreover, in the case of underdamped systems, also check that the obtained responses correspond to the values computed at point (c).

Solution: (a) The poles of the transfer functions are calculated as follows:

$$\begin{aligned} H_1(s) &: s^2 + 1.6s + 16 = 0 \quad \Rightarrow \quad s_{1,2} = -0.8 \pm 3.919j \\ H_2(s) &: 2s^2 + 8s + 8 = 0 \quad \Rightarrow \quad s_{1,2} = -2 \\ H_3(s) &: s^2 + 7s + 10 = 0 \quad \Rightarrow \quad s_{1,2} = \{-2, -5\} \end{aligned}$$

(b) In order to find $H(0)$, ω_0, ζ, the transfer functions must be in one of the following two normalized forms:

$$H(s) = \frac{H(0)\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{H(0)}{\frac{s^2}{\omega_0^2} + \frac{2\zeta}{\omega_0}s + 1}$$

For $H_1(s)$, we obtain:

$$H_1(s) = \frac{15}{s^2 + 1.6s + 16} \Rightarrow \begin{cases} H(0)\omega_0^2 = 15 \\ 2\zeta\omega_0 = 1.6 \\ \omega_0^2 = 16 \end{cases} \Rightarrow \begin{cases} \omega_0 = 4\,\text{rad/s} \\ H(0) = 15/16 = 0.9375 \\ \zeta = 1.6/2\omega_0 = 0.2 \end{cases}$$

The process represented by $H_1(s)$ is underdamped since $0 < \zeta < 1$.

We can rewrite $H_2(s)$ as:

$$H_2(s) = \frac{5/2}{s^2 + 4s + 4} = \frac{5/8}{1 + s + s^2/4}$$

from which we obtain:

$$\begin{cases} H(0)\omega_0^2 = 5/2 \\ 2\zeta\omega_0 = 4 \\ \omega_0^2 = 4 \end{cases} \Rightarrow \begin{cases} \omega_0 = 2\,\text{rad/s} \\ \zeta = 4/2\omega_0 = 1 \\ H(0) = 5/2\omega_0^2 = 5/8 = 0.6250 \end{cases}$$

The process represented by $H_2(s)$ is critically damped since $\zeta = 1$.

For $H_3(s)$, we obtain:

$$H_3(s) = \frac{10}{s^2 + 7s + 10} \Rightarrow \begin{cases} H(0)\omega_0^2 = 10 \\ 2\zeta\omega_0 = 7 \\ \omega_0^2 = 10 \end{cases} \Rightarrow \begin{cases} \omega_0 = \sqrt{10}\,\text{rad/s} \approxeq 3.16\,\text{rad/s} \\ \zeta = 7/2\sqrt{10} \approxeq 1.11 \\ H(0) = 10/\omega_0^2 = 1 \end{cases}$$

The process represented by $H_3(s)$ is overdamped since $\zeta > 1$.

(c) For underdamped second-order systems, the maximum overshoot can be computed as follows:

$$MO[\%] = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

The 5% and 2% settling times are computed as follows:

$$T_{s5\%} = \frac{3}{\zeta\omega_0} \qquad T_{s2\%} = \frac{4}{\zeta\omega_0}$$

The peak time and rise time are computed as:

$$t_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} \quad t_r = \frac{1}{\omega_0\sqrt{1-\zeta^2}}\left[\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

Given the previously computed ω_0 and ζ, we must compute the above values for $H_1(s)$, thus obtaining Table 5.1.

Transfer function	MO [%]	$T_{s5\%}$	$T_{s2\%}$	t_p	t_r
$H_1(s)$	52.66	3.75 s	5 s	0.8 s	0.45 s

Table 5.1: Second-order response parameters.

(d) Using the final value theorem, we can compute:

$$y_1(\infty) = \lim_{s\to 0} s\frac{H_1(s)}{s} = H_1(0) = 0.9375$$
$$y_2(\infty) = \lim_{s\to 0} s\frac{H_2(s)}{s} = H_2(0) = 0.6250$$
$$y_3(\infty) = \lim_{s\to 0} s\frac{H_3(s)}{s} = H_3(0) = 1$$

(e) The step responses are shown in Figs. 5.2-5.4. It can be seen that they are compatible with the values computed at points (c)-(d) of this exercise.

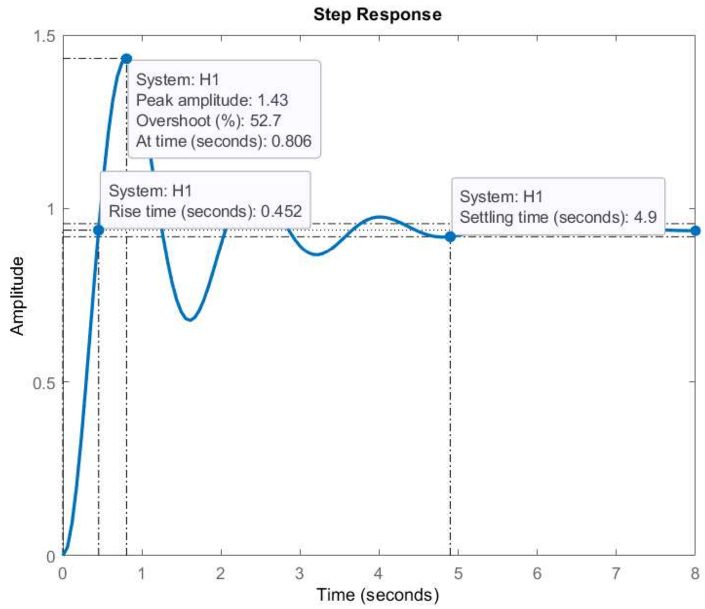

Figure 5.2: Step response $y_1(t)$ (Second-order).

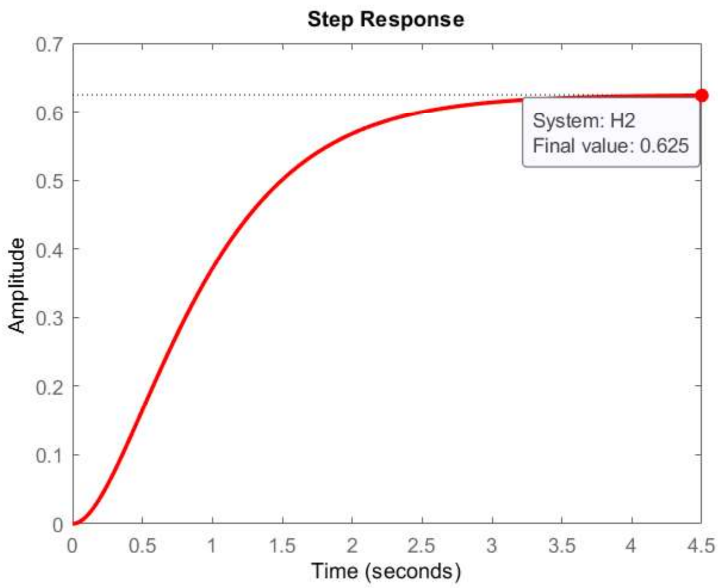

Figure 5.3: Step response $y_2(t)$ (Second-order).

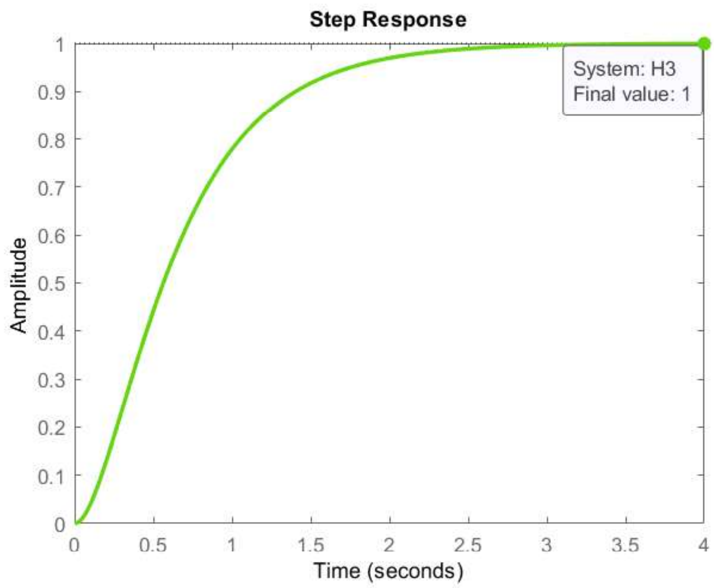

Figure 5.4: Step response $y_3(t)$ (Second-order).

5.3 Hob (NO: kokeplate) (mandatory)

Let's consider a hob (på norsk: kokeplate), as those you can find in your home. The following equation describes the typical dynamics of the temperature $T(t)$:

$$\dot{T}(t) = \frac{P(t)}{mc_p} - \frac{hA}{mc_p}\left(T(t) - T_{amb}(t)\right) \tag{5.6}$$

where:

- $P(t)$ [W] is the electrical power applied to the hob;
- $T(t)$ [°C] is the temperature of the hob, assumed to be evenly distributed;
- $T_{amb}(t) = 20°\text{C}$ is the ambient temperature in the kitchen;
- $m = 0.5\,\text{kg}$ is the mass;
- $c_p = 460\,\text{J}/(\text{kg}°\text{C})$ is the specific heat capacity of the hob;
- $h = 20\,\text{J}/(\text{sm}^2\,°\text{C})$ is the specific heat transfer coefficient between the hob and the air;
- $A = 0.1\,\text{m}^2$ is the area of the hob/air surface.

In the following, we will consider that:

- $T(t)$ is the output variable $y(t) = T(t)$;

- $P(t)$ is the input variable $u(t) = P(t)$;
- $T_{amb}(t)$ is the disturbance signal $w(t) = T_{amb}(t)$.

Question 5.3

Do the following:

(a) Laplace transform (5.6) and find the transfer function between $U(s)$ and $Y(s)$ as:

$$H_p(s) = \frac{Y(s)}{U(s)} = \frac{H_p(0)}{1 + s\tau} \tag{5.7}$$

What are the values of $H_p(0)$ and τ?

(b) Find the transfer function from $W(s)$ to $Y(s)$ as:

$$H_w(s) = \frac{Y(s)}{W(s)} = \frac{H_w(0)}{1 + s\tau_w} \tag{5.8}$$

What are the values of $H_w(0)$ and τ_w? Explain why $H_w(0) = 1$.

(c) Find the poles of $H_p(s)$. Confirm the result by using the `pzmap` command in MATLAB.

Solution: (a) Eq. (5.6) can be expressed as:

$$\dot{y}(t) = \frac{1}{mc_p}u(t) - \frac{hA}{mc_p}\left(y(t) - w(t)\right)$$

which is Laplace transformed into:

$$sY(s) = \frac{1}{mc_p}U(s) - \frac{hA}{mc_p}\left(Y(s) - W(s)\right)$$

that can be rewritten as:

$$\left(s + \frac{hA}{mc_p}\right)Y(s) = \frac{1}{mc_p}U(s) + \frac{hA}{mc_p}W(s) \tag{5.9}$$

By setting $W(s) = 0$ we can find $H_p(s) = Y(s)/U(s)$ as:

$$H_p(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{mc_p}}{s + \frac{hA}{mc_p}} = \frac{\frac{1}{hA}}{1 + s\frac{mc_p}{hA}}$$

By replacing appropriately the parameters' values, we find:

$$H_p(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{20\cdot 0.1}}{1 + s\frac{0.5\cdot 460}{20\cdot 0.1}} = \frac{0.5}{1 + s115} \tag{5.10}$$

Hence, the static gain is $H_p(0) = 0.5$ whereas the time constant is $\tau = 115$ s.

(b) By setting $U(s) = 0$ in (5.9):

$$\left(s + \frac{hA}{mc_p}\right) Y(s) = \frac{hA}{mc_p} W(s)$$

so that:

$$H_w(s) = \frac{Y(s)}{W(s)} = \frac{\frac{hA}{mc_p}}{s + \frac{hA}{mc_p}} = \frac{1}{1 + s\frac{mc_p}{hA}}$$

By replacing the parameters' values, we obtain:

$$H_w(s) = \frac{Y(s)}{W(s)} = \frac{1}{1 + s\frac{0.5 \cdot 460}{20 \cdot 0.1}} = \frac{1}{1 + 115s}$$

The static gain is $H_w(0) = 1$ and the time constant is $\tau_w = 115$ s. The reason why $H_w(0) = 1$ is that if we assume $u(t) = 0$, then the hob's temperature $T(t)$ will converge to $T_{amb}(t)$ at steady-state.

(c) The pole of $H_p(s)$ can be found by setting the denominator of $H_p(s)$ to zero:

$$115s + 1 = 0 \quad \Rightarrow \quad s = p = -\frac{1}{\tau} = -\frac{1}{115}$$

In MATLAB you can use the following lines (see Fig. 5.5):

```
>> H=tf(0.5,[115 1])

H =

0.5
---------
115 s + 1

Continuous-time transfer function.

>> pzmap(H)
>> axis([-0.02 0 -1 1])
```

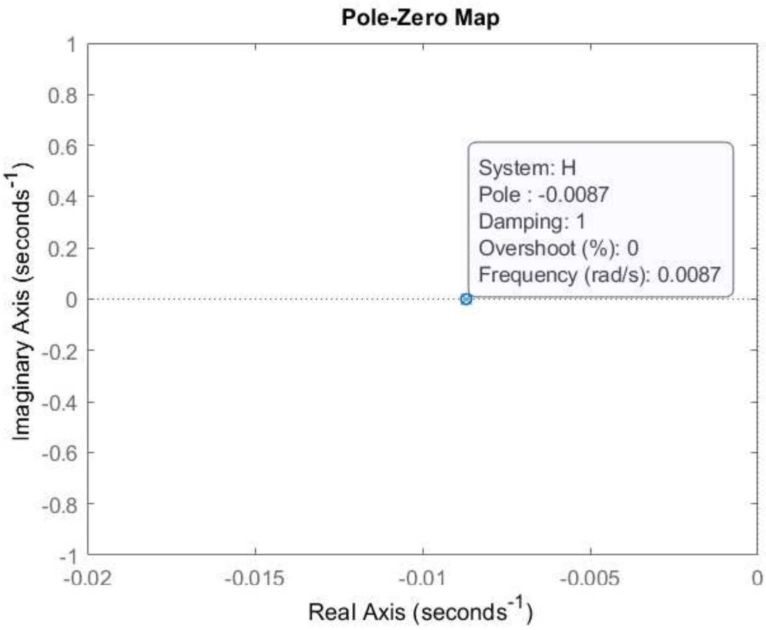

Figure 5.5: The transfer function $H_p(s)$ has a single pole in $-1/115 = -0.0087$, as calculated.

To simulate the hob, we will use the Simulink model `kokeplate_tidskonstant.slx` which is shown in Fig. 5.6

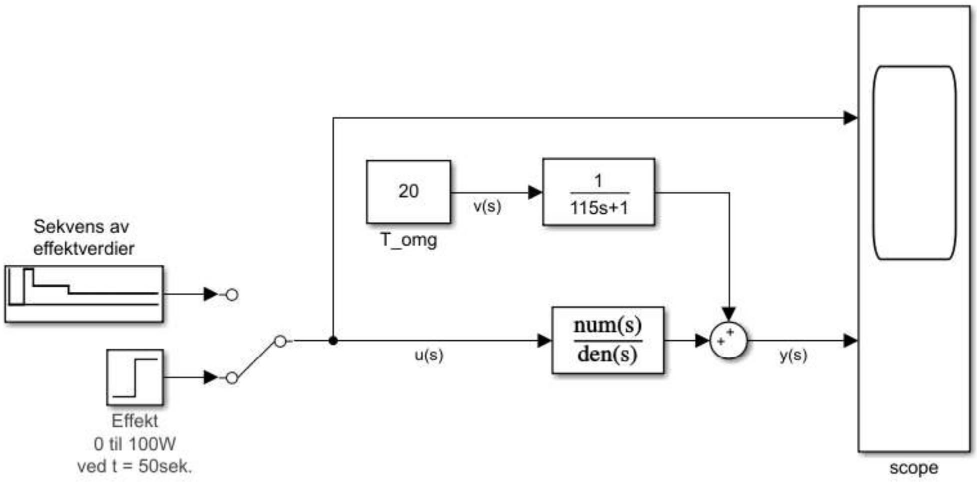

Figure 5.6: Simulink model of the hob.

To avoid an initial transient response from $0°C$ to $20°C$ due to the ambient temperature (before turning on the power), Simulink has a block called `Transfer Fcn (with initial output)` which you can find in the Simulink Extras folder under Additional Linear. Therein, we specify the system to have an initial output of 20 (same as the constant block which represents $T_{omg}(t)$). Double-click on the block to see how this is specified.

The model uses the following simulation configuration parameters:

- step time: 500 s
- integration method: Euler
- fixed step length: 1 s

By simulating the model, you apply 100 W at $t = 50$ s (this corresponds to set 1 on the switch). Remember that the total response consists of the sum of the partial responses due to the ambient temperature and the input:

$$Y(s) = H_p(s)U(s) + H_w(s)W(s) \tag{5.11}$$

Question 5.4

Based on the response in Simulink, determine the static gain $H_p(0)$ and the time constant τ for the transfer function $H_p(s)$. You can do this using the built-in Cursor Measurement function in Simulink, see Fig. 5.7.

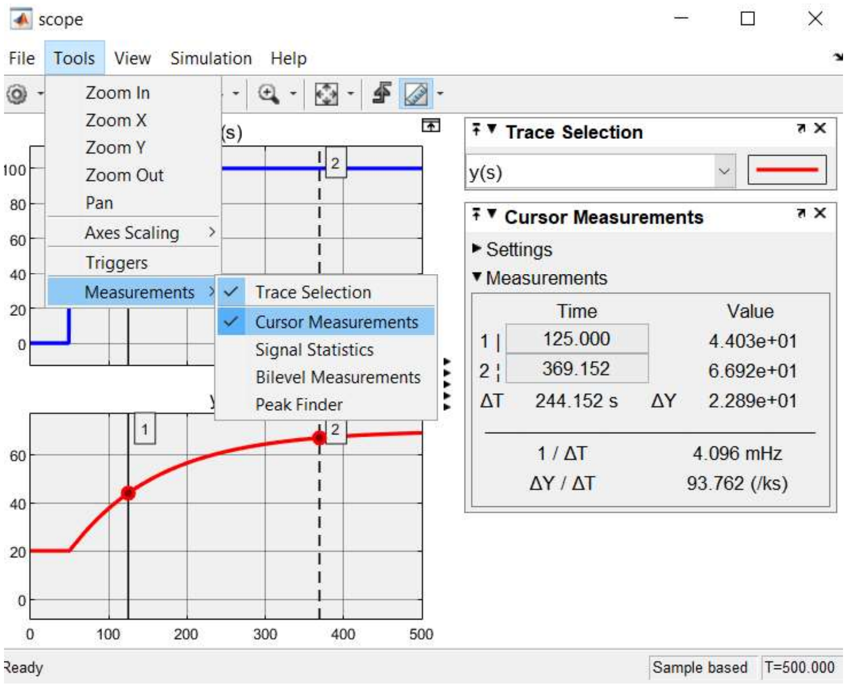

Figure 5.7: Tool for reading response data.

Solution: By setting the hob to 100 W, the output moves to 70°C. The starting point was 20°, which means that the 100 W step input produces an increase of the output of 50°C. This is confirmed by taking the product of the input magnitude (100 W) and the static gain $H_p(0) = 0.5$, see Eq. (5.10). To confirm the time constant τ, we take 63% of 50°C which is 31.5°C. The time constant is therefore read 31.5°C above the starting point of 20°C, i.e. 51.5°C. In addition, we must take into account that the step input was applied at $t = 50$ s, which must be subtracted from time.

By using the reading tool in Simulink we find the data in Fig. 5.8.

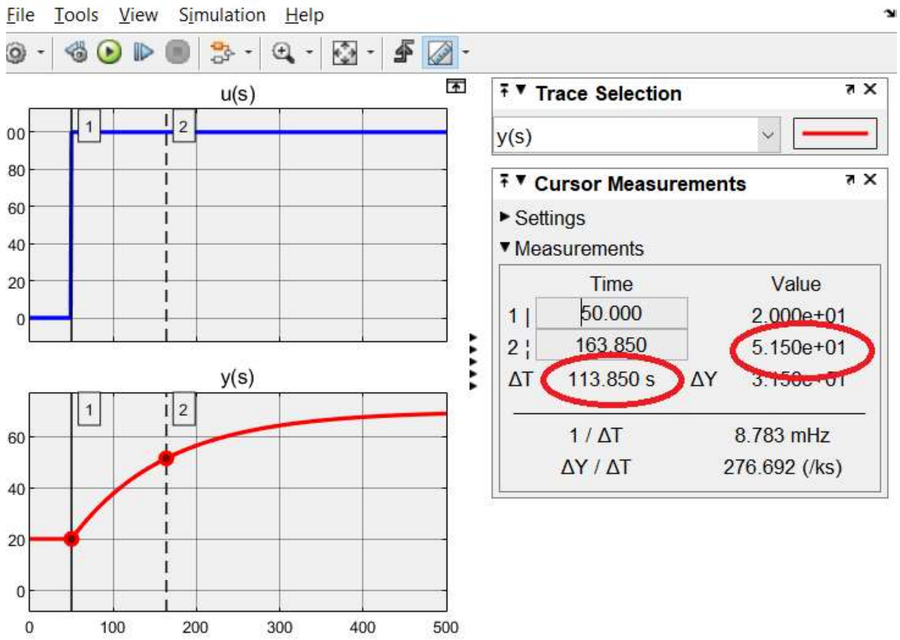

Figure 5.8: Tool for reading response data.

We see that the response passes to 51.5°C after t = 113.85 s, which is very close to the value τ = 115 s appearing in Eq. (5.10).

Question 5.5

Let us assume that you think that the time constant was relatively long when you turned the hob to 1, and that you would rather examine the time constant when you set the hob to 2, which is equivalent to 200 W. Simulate the model with this new input and determine the gain and time constant. Have they changed, and if so, to what value? What has this taught you?

Solution: By using 200 W, the output moves to 120 degrees. The starting point was 20 degrees, which gives an increase of 100 degrees. The 63% of 100 degrees is 63 degrees, so the time constant will be read 63 degrees above the starting point of 20 degrees, i.e. at 83 degrees. In addition, the step time t = 50 seconds must be subtracted from the value read on the plot. Using the reading tool in Simulink, we find the data shown in Fig. 5.9.

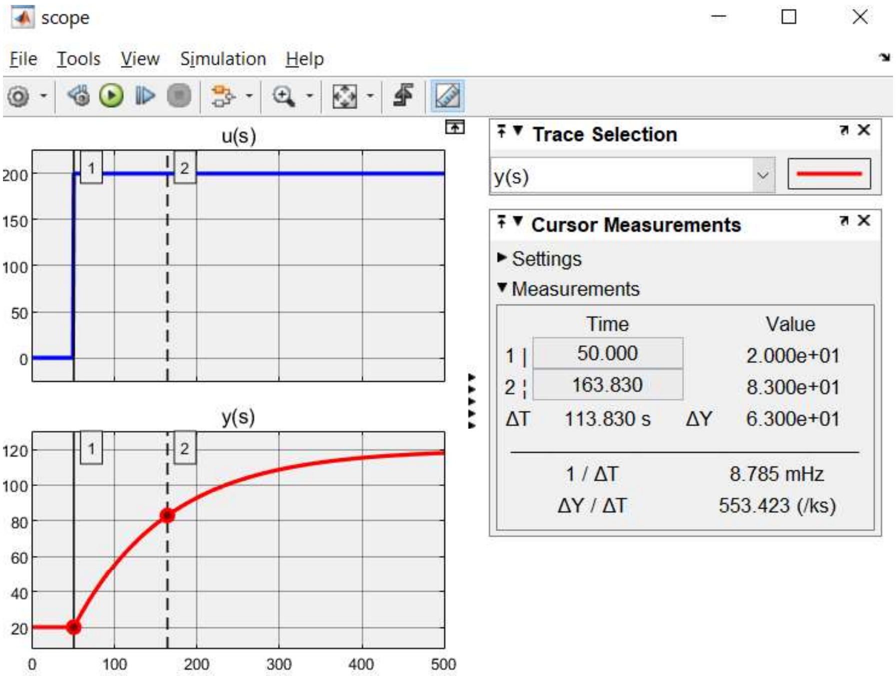

Figure 5.9: Reading the time constant in Simulink.

You can see that the output reaches 83 degrees after $\tau = 113.8$ seconds. This is identical to the value found previously and quite close to the actual value of $\tau = 115\,s$ that we found in $H_p(s)$. This tells us that the time constant is independent from the size of the input signal: you do not change the dynamical properties of the hob[1] ($H(0)$ and τ) by increasing the size of the input signal. However, the hob will reach a higher temperature.

As a chef, how would you get the hob to achieve a temperature of 120°C as soon as possible (without overshoot)? The hob can be turned up to power 6, i.e., 600 W. To find out this, double-click on the Switch so that you use the block `Sekvens av effektverdier` (this is a Source block called Repeating Sequence Stair). By double-clicking on the block you get the window shown in Fig. 5.10.

[1]This statement holds true as long as the linearized model is still a valid approximation of the underlying nonlinear system!!

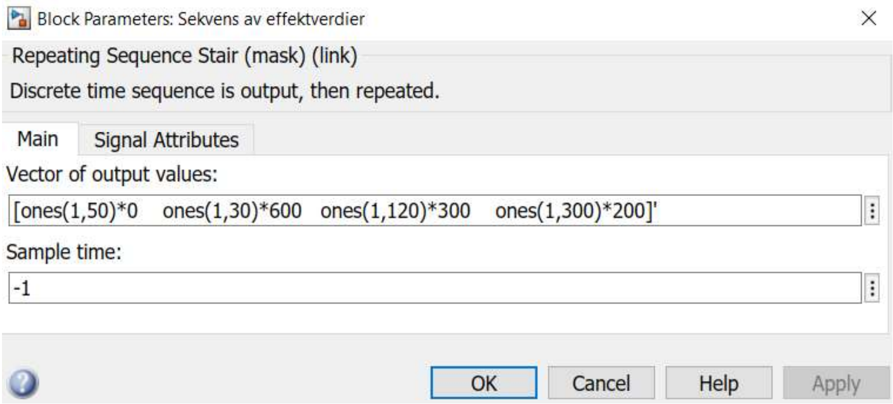

Figure 5.10: Parameterization of the hob input.

In Fig. 5.10, 0 W is used for the first 50 seconds, followed by 600 W for 30 seconds, then 300 W for 120 seconds and finally 200 W for 300 seconds. Make sure that the sum of the lengths of the `ones` is the same as the simulation time (500), e.g., 50+30+120+300 = 500. Always use 0 W in the first 50 seconds so that the first step input happens at 50 seconds.

Question 5.6

Your task is to find the sequence of input values that makes the time required by the hob to reach 120°C as short as possible (remember that the temperature should not exceed 120°C either). In this way, the response time T_r (i.e., the time required to reach 63% of the final value using active manipulation of the control signal) will be as short as possible, and you would act like a human temperature controller for the hob. What value do you find for the response time T_r when you choose actively the input signal? How much smaller is it when compared to the open-loop time constant?

Comment: As you have now learned, the open-loop time constant tells you how the system reacts to a step input of arbitrary magnitude. However, applying a step input is not the most efficient way to get a system to swing to a new value of its output: it is possible to make a system to behave faster by using actively the input signal, which is exactly what the controller will do *automatically* in a feedback control system.

Solution: After some trial and error with the Repeating Sequence Stair, the start-up sequence shown in Fig. 5.11 was found, which gives the response shown in Fig. 5.12.

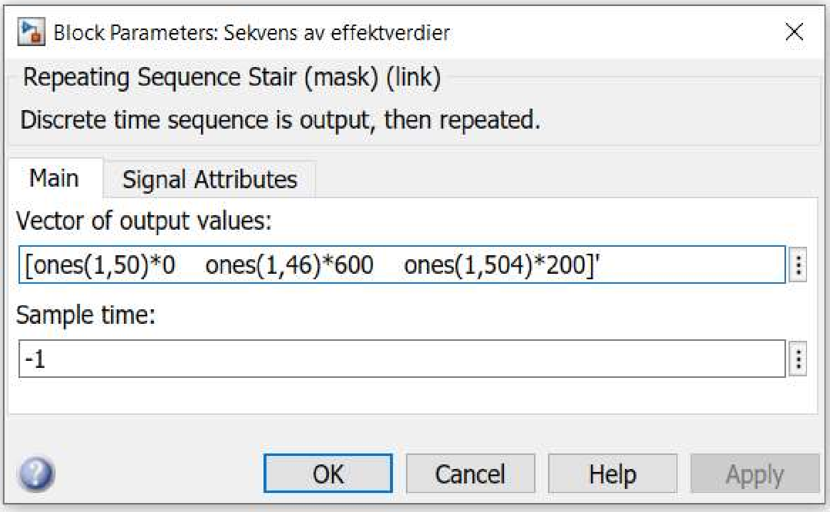

Figure 5.11: Input sequence to drive the hob to 115 degrees much faster.

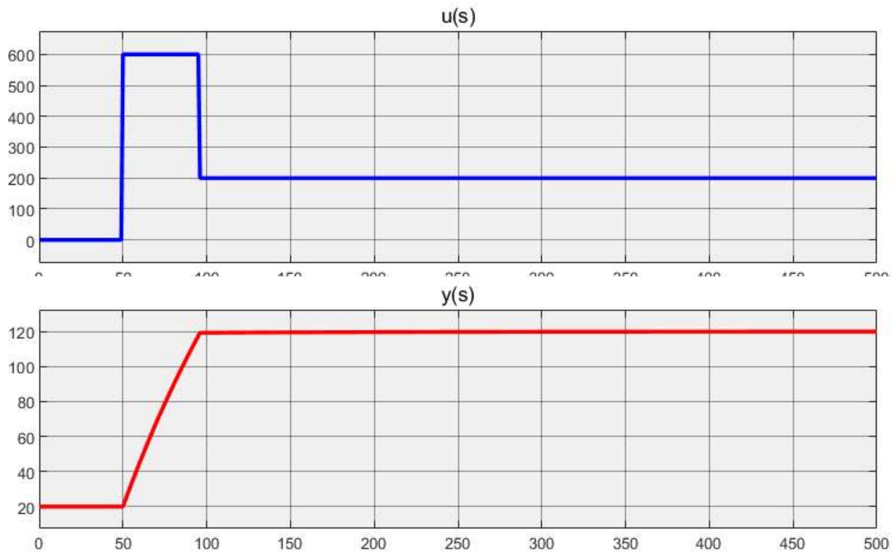

Figure 5.12: Response with active use of the control input.

We see (Fig. 5.13) that the output reaches 83 degrees at 76.99 seconds, which corresponds to an equivalent time constant of:

$$T_r = 26.99 \text{ seconds}$$

which is more than 4 times faster than the open-loop process.

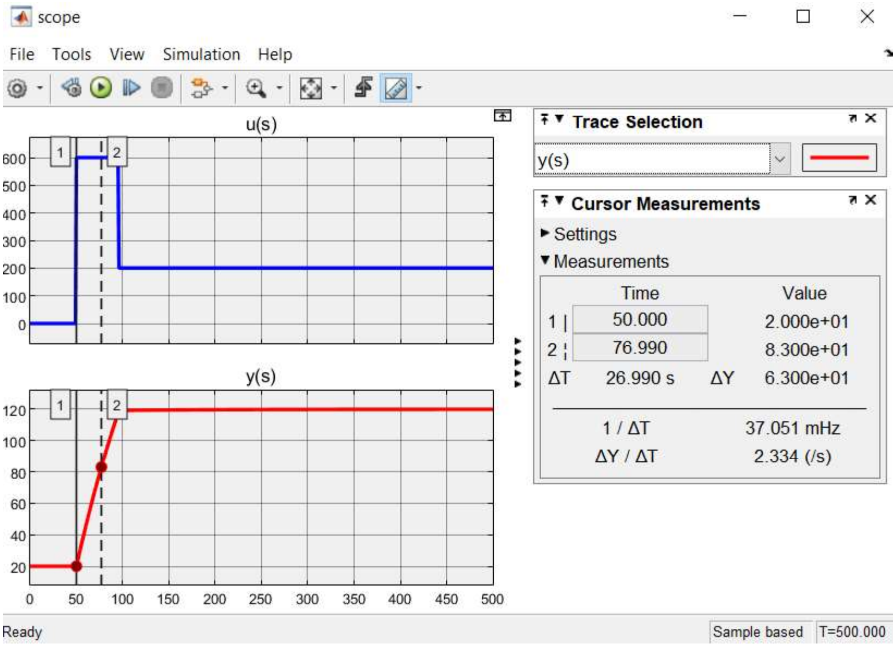

Figure 5.13: Response with active use of the control input.