6 Assignment

6.1 Frequency response

Let us consider the following first-order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2}{1+4s} \tag{6.1}$$

Question 6.1

Find the frequency response $H(j\omega)$, and decompose it into the magnitude $|H(j\omega)|$ and phase $\arg(H(j\omega))$.

Solution: By setting $s = j\omega$ in $H(s)$, we obtain:

$$H(j\omega) = \frac{2}{\sqrt{1+16\omega^2}} e^{j \arctan 4\omega} = \frac{2}{\sqrt{1+16\omega^2}} e^{-j \arctan 4\omega}$$

which means that:

$$|H(j\omega)| = \frac{2}{\sqrt{1+16\omega^2}}$$
$$\arg(H(j\omega)) = -\arctan 4\omega$$

Question 6.2

What is the steady-state response (stationary oscillations) of $y(t)$ when $u(t) = 0.5\sin(0.6t)$? Hint: use the result from Question 6.1.

Solution: First, we must calculate:

$$|H(0.6j)| = \frac{2}{\sqrt{1+16\cdot 0.6^2}} = 0.77$$
$$\arg(H(0.6j)) = -\arctan(4\cdot 0.6) = -1.18\,\text{rad}$$

which mean that:

$$y(t) = 0.5\cdot 0.77\sin(0.6t-1.18) = 0.385\sin(0.6t-1.18)$$

Question 6.3

Verify the result at the previous point using Simulink (you will need a Sine Wave, a Transfer Fcn and a Scope block; remember to specify the correct frequency and amplitude in the sinusoidal signal. Also, use a total length of the simulation equal to 50 s, and by right-clicking on a blank point in the screen and selecting Model Configuration Parameters, then Solver Details, change the Max Step Size to 0.01 in order to obtain more precise simulations). You must debate appropriately that the response computed in Question 6.2 matches the simulated response. You can use the Cursor Measurements function in the Tools menu to do so.

Solution: The simulation results are shown in Fig. 7.1. By reading the maximum value of the steady-state sine wave, you can see that it reaches 0.385, as expected from the calculations.

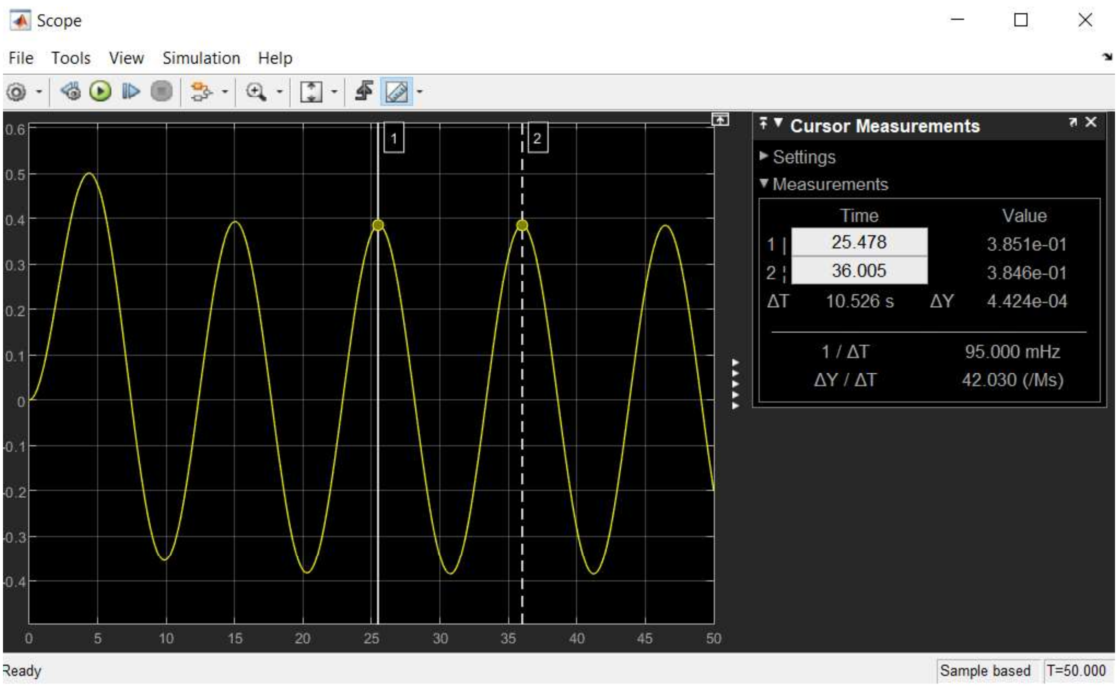

Figure 6.1: Scope with Cursor Measurements.

To calculate the phase, we need the period T, which is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.6} = 10.47 \text{ s}$$

Then, from the figure we can see that the steady-state peaks occur at ..., 25.48, 36.00, ... which taking into account the period T leads to computing previous peak times as 15.01 s and 4.54 s.

Then, it must be necessarily:

$$\sin(0.6 \cdot 4.54 + \phi) = 1 \quad \rightarrow \quad 0.6 \cdot 4.54 + \phi = 2.72 + \phi = \pi/2 = 1.57 \quad \rightarrow \phi = 1.57 - 2.72 = -1.15$$

which is very close to the computed value -1.18.

On Canvas, you will find a .m file named `manuell_bodeplot.m` in which you should correct the lines 19, 24, 30 and 35 (in the provided file, everything is set to be equal to zero, which is not the correct way to compute the frequency response of a transfer function $H(s) \neq 0$. You can use your answer to Question 6.1 to correct appropriately the script). Pay attention to using the correct logarithm command to fill line 30 (write `help log` and `help log10` in MATLAB's Command Window to check how to use the available logarithm commands).

Different graphical representation of the frequency response should appear (if the figure does not fit your screen, then write in the Command Window `get(gcf,'position')` and replace appropriately the four numbers that you find at line 39 of the script, then run it again):

- The first column shows the magnitude and the phase (in degrees) using a linear frequency axis
- The second column shows the magnitude and the phase (in degrees) using a logarithmic frequency axis
- The third column shows the magnitude (in dB) and the phase (in degrees) using a logarithmic frequency axis
- The fourth column shows the magnitude (in dB) and the phase (in degrees) using a logarithmic frequency axis
- The fifth column shows the `bode` command, hence the magnitude (in dB) and the phase (in degrees) are represented using a logarithmic frequency axis (as in the fourth column)

The frequency ω varies between 0.01 rad/s and 10 rad/s. The purpose of this exercise is to study the differences in the graphical representation of the frequency response depending on whether the x or the y axis are linear or logarithmic. The information given in the plots is always the same, it is only a matter of how the information is represented.

Question 6.4

Use the Bode plot (column 5) to read the magnitude (in dB) and the phase of the frequency response for the following frequencies:

- 0.1 rad/s
- 0.6 rad/s
- 1.0 rad/s

Click first on the icon Data Cursor and then click on the lines in the plot so that you get a square cursor that you can move back and forth using the arrow keys of your keyboard.

> To add a new point to the same curve, you can press the Shift key while clicking on the line. Attach a screenshot of the figure with all the relevant information. Calculate using what you read from the plot what is the amplification/attenuation and the frequency shift that we would expect for sinusoidal inputs at these specific frequencies.

Solution: The lines in the MATLAB code are corrected as follows:

```
for i = 1:max(size(omega))
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% amplitudeforsterkning som ren forsterkning
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ampl(i) =  2/sqrt((1 + 16*omega(i)^2));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% fase i radianer
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fase_rad(i) = -atan(4*omega(i));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% amplitudeforsterkning i dB (beregnet fra ampl)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ampl_dB    = 20*log10(ampl);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% fase i grader (beregnet fra fase_rad)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fase_grader =  fase_rad*360/(2*pi);
```

By running the code, Fig. 6.2 is obtained.

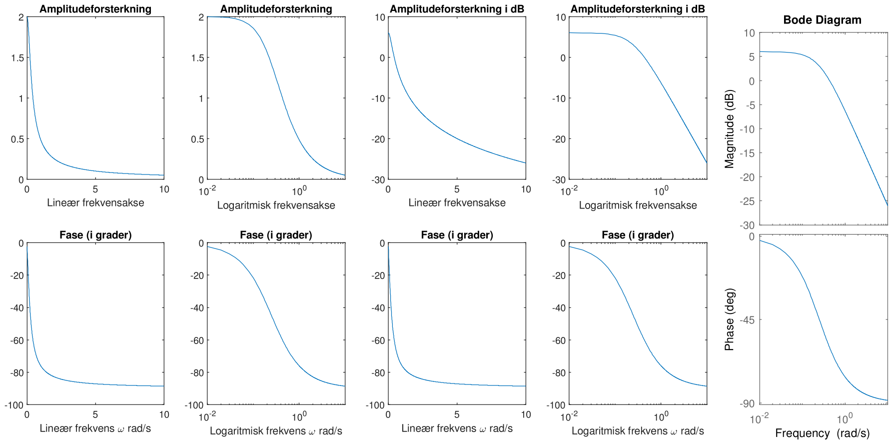

Figure 6.2: Different graphical representations of the frequency response.

Then, by using the Data cursor on the Bode diagram (see Fig. 6.3), the following can be obtained:

- $\omega = 0.1\,\text{rad/s}$: $|H(0.1j)|_{\text{dB}} = 5.38\,\text{dB} \Rightarrow |H(0.1j)| = 10^{5.38/20} = 1.8570$, $\arg(H(0.1j)) = -21.8$ degrees
- $\omega = 0.6\,\text{rad/s}$: $|H(0.6j)|_{\text{dB}} = -2.28\,\text{dB} \Rightarrow |H(0.6j)| = 10^{-2.28/20} = 0.7691$, $\arg(H(0.6j)) = -67.4$ degrees
- $\omega = 1\,\text{rad/s}$: $|H(j)|_{\text{dB}} = -6.28\,\text{dB} \Rightarrow |H(j)| = 10^{-6.28/20} = 0.4853$, $\arg(H(j)) = -76$ degrees

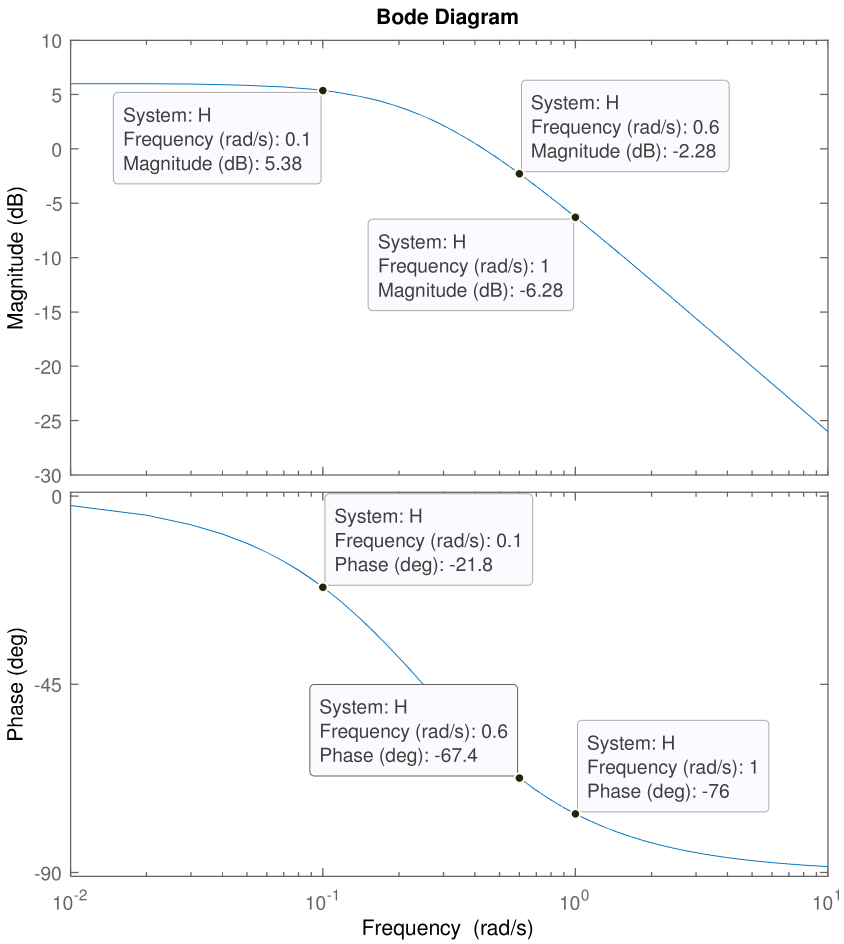

Figure 6.3: Reading of the Bode diagram at the three different frequencies.

Question 6.5

Use column 2 and read the magnitude and phase of the frequency response at the same frequencies as in Question 6.4. Do the values correspond to those that you would calculate using the analytic expression of the frequency response found in Question 6.1? Attach a relevant screenshot of the figure with the readings.

Solution: The values shown in Fig. 6.4 correspond to those found in the previous question.

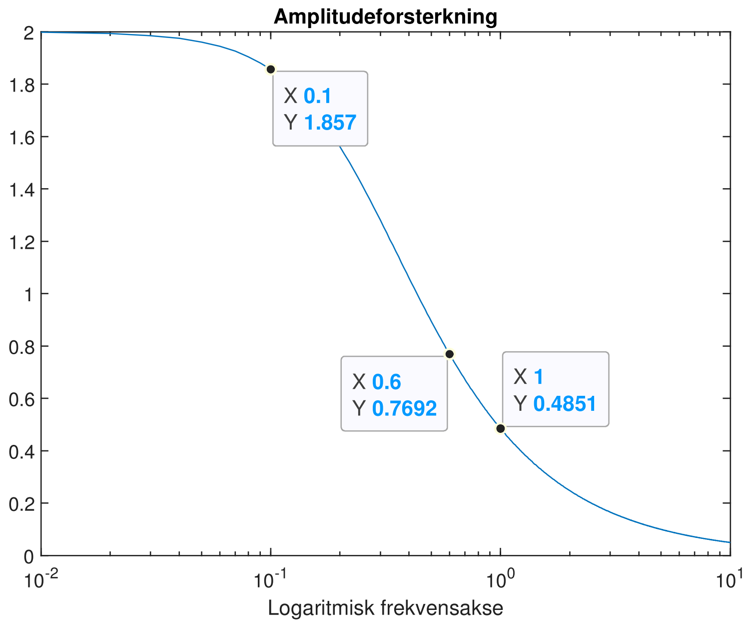

Figure 6.4: Reading of $|H(j\omega)$ at the three different frequencies.

Question 6.6

Use Simulink to simulate $H(s)$ with the input $u(t) = \sin(t)$ (use the same settings as in Question 6.3 to get more precise simulations). Visualize the results in a Scope, and compute from it the amplitude and the phase shift. Do they correspond to the amplitude/phase computed as an answer to Question 6.4?

Solution: The results of the simulation are shown in Fig. 6.5. It can be seen that the amplitudes of the steady-state peaks are at 0.4852, which is very close to the predicted values. The period T is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 6.28 \text{ s}$$

Then, from the figure we can see that the steady-state peaks occur at ..., 28.07, 34.36, ... which, taking into account T, means that previous peaks occurred at 21.79, 15.51, 9.23 and 2.95 seconds. Then, it must necessarily be:

$$\sin(2.95 + \phi) = 1 \quad \longrightarrow \quad 2.95 + \phi = \pi/2 = 1.57 \quad \longrightarrow \phi = 1.57 - 2.95 = -1.38 \,\text{rad}$$

To convert radiant to degrees, we multiply by 180 and divide by π:

$$-1.38 \frac{180}{\pi} = -79 \,\text{degrees}$$

which is very close to the theoretical computed value.

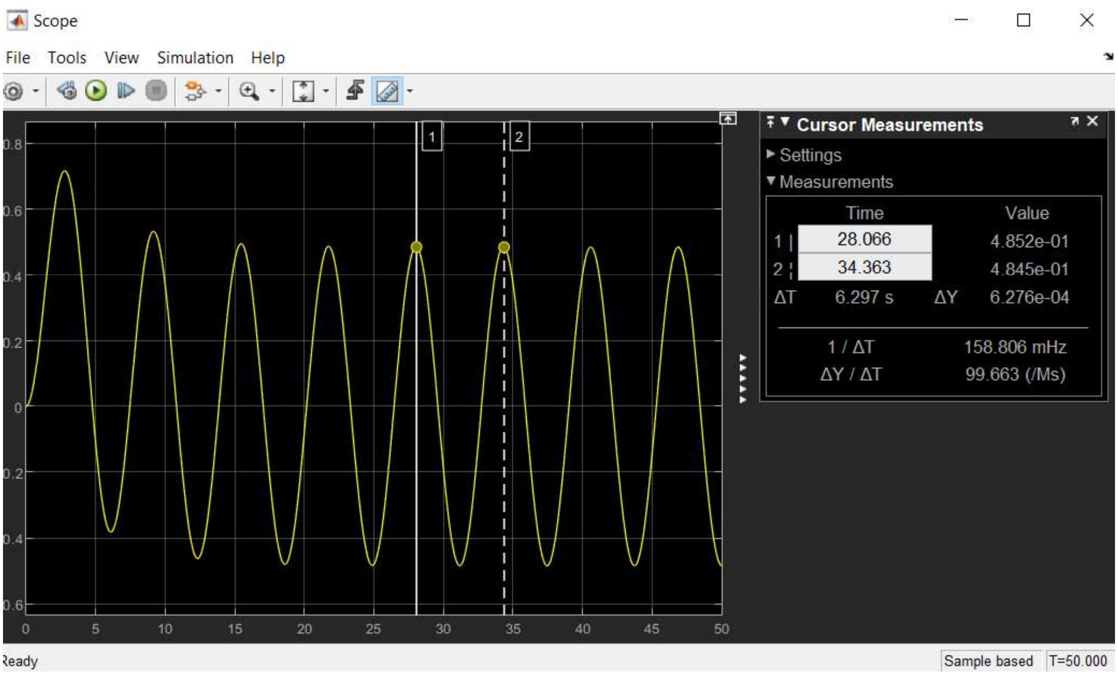

Figure 6.5: Simulation of $H(s)$ with the input $u(t) = \sin(t)$.

6.2 Bode diagrams and lead-lag compensation

Lead compensators and lag compensators are components in a control system that improve an undesirable frequency response in a feedback control system. They both introduce a pole-zero pair into the open-loop transfer function, and are defined as:

$$H(s) = \frac{p}{z}\frac{s+z}{s+p} = \frac{1+s/z}{1+s/p} \tag{6.2}$$

with $p > 0$ and $z > 0$.

In a lead compensator $z < p$; then, the introduction of $H(s)$ in the direct loop has the effect of raising the amplitude and the phase plots, as shown for example in Fig. 6.6 (remember that multiplications between transfer functions become additions when looking at Bode diagrams, due to the multiplication property of the logarithm).

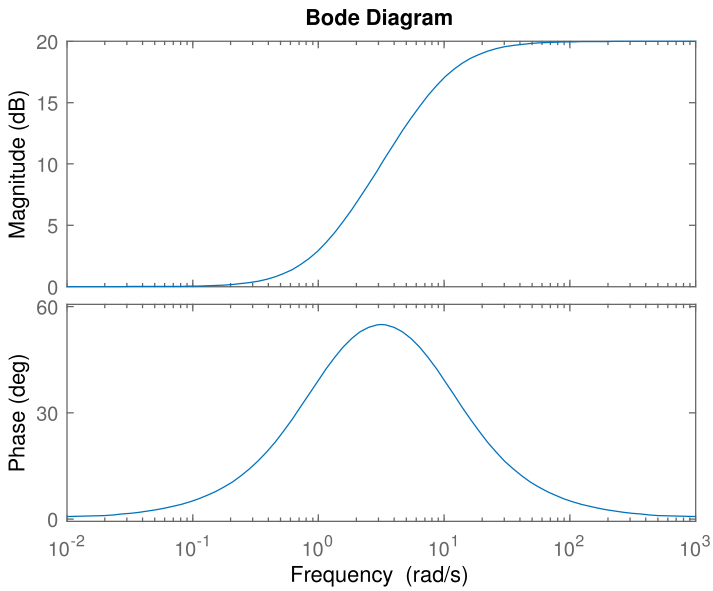

Figure 6.6: Bode diagram of the lead compensator $H(s) = 10(s + 1)/(s + 10)$.

On the other hand, in a lag compensator $z > p$; then, the introduction of $H(s)$ in the direct loop has the effect of lowering the amplitude and the phase plots, as shown in Fig. 6.7.

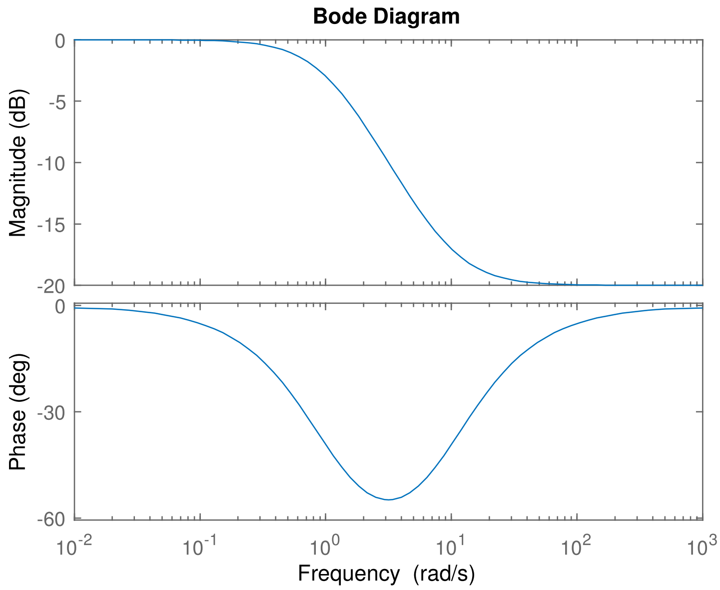

Figure 6.7: Bode diagram of the lead compensator $H(s) = (s + 10)/10(s + 1)$.

Question 6.7

Given the following transfer functions:

$$H_1(s) = \frac{0.25}{s(s+1)^2} \qquad H_2(s) = \frac{25(s+1)}{20s^2 + 5s + 5} \tag{6.3}$$

design a lead-lag compensator $H_c(s)$ (combination of gain factors, lead compensators and lag compensators) such that the series interconnection of $H_c(s)$ and either one between $H_1(s)$ and $H_2(s)$ raises the corresponding Bode amplitude plots by 20 dB in the ranges of frequencies $\omega < 1\text{rad/s}$ and $\omega > 10\text{rad/s}$. Using the MATLAB command `bode` the Bode diagram of the initial transfer functions $H_1(s)$ and $H_2(s)$ with the respective series interconnections with $H_c(s)$, i.e., $H_1(s)H_c(s)$ and $H_2(s)H_c(s)$, respectively (you may use the command `series` for creating a series interconnection of transfer functions). *Hint: it is impossible to design a transfer function $H_c(s)$ that achieves perfectly the goal in this question. A more realistic objective would be to obtain a transfer function $H_c(s)$ whose asymptotic Bode diagram looks as in Fig. 6.8. Alternatively, by moving appropriately the corner frequencies, you might obtain a transfer function whose asymptotic Bode diagram looks as in Fig. 6.9. Note that there is not a single way to solve this exercise, and different solutions might be correct, although different solutions might be better/worse.*

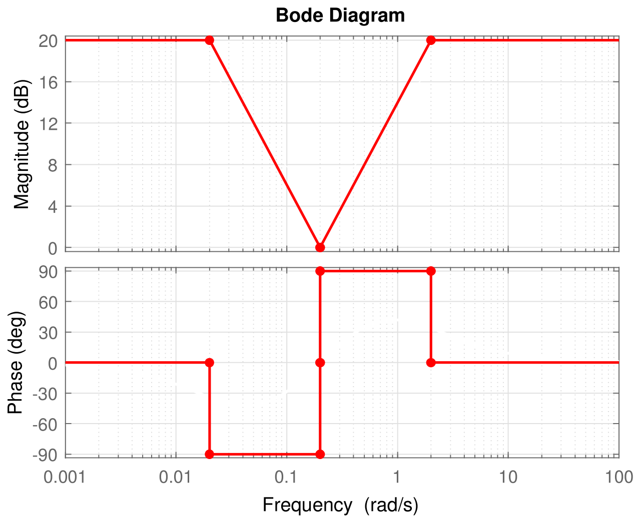

Figure 6.8: Hint asymptotic Bode diagram of $H_c(s)$.

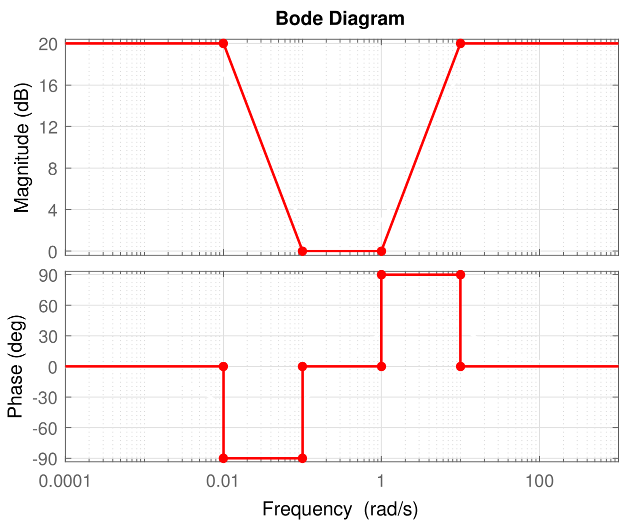

Figure 6.9: Hint asymptotic Bode diagram of $H_c(s)$ (ii).

Solution: The asymptotic Bode diagrams shown in Figs. 6.8-6.9 correspond to:

$$H_c^1(s) = 10\frac{1+5s}{1+0.5s}\frac{1+5s}{1+50s} = \frac{500s^2+200s+20}{50s^2+101s+2}$$
$$H_c^2(s) = 10\frac{1+s}{1+10s}\frac{1+10s}{1+100s} = \frac{10s^2+11s+1}{s^2+10.01s+0.1}$$

The following lines of code are used to generate the Bode diagrams shown in Figs. 6.10-6.11

(note that the original Bode diagrams are raised by 20dB in the prescribed ranges of frequencies).

```
>> H1 = tf([0.25],[1 2 1 0]);
>> H2 = tf([25 25],[20 5 5]);
>> Hc1 = tf([500 200 20],[50 101 2]);
>> Hc2 = tf([10 11 1],[1 10.01 0.1]);
>> bode(H1,series(Hc1,H1),series(Hc2,H1));
>> legend('H_1(s)','H_c^1(s)H_1(s)','H_c^2(s)H_1(s)');
>> bode(H2,series(Hc1,H2),series(Hc2,H2));
>> legend('H_2(s)','H_c^1(s)H_2(s)','H_c^2(s)H_2(s)');
```

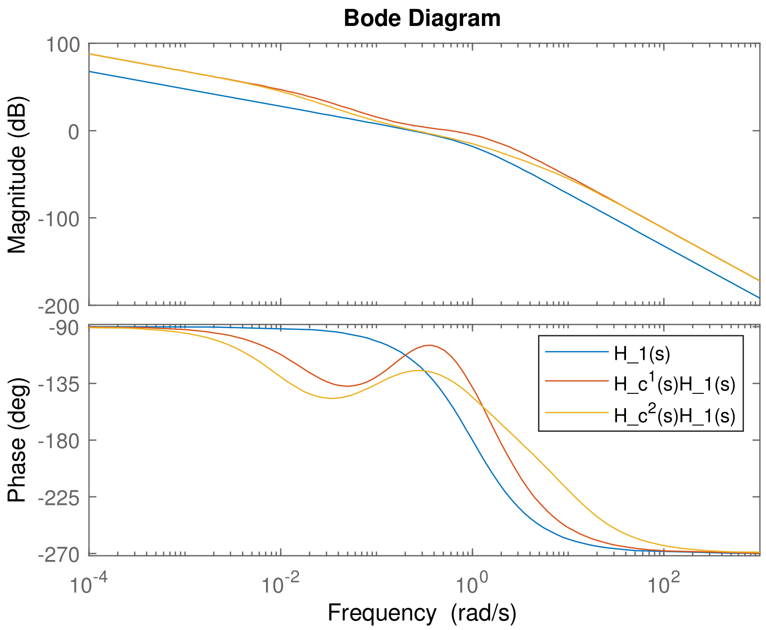

Figure 6.10: Lead-lag compensation for $H_1(s)$.

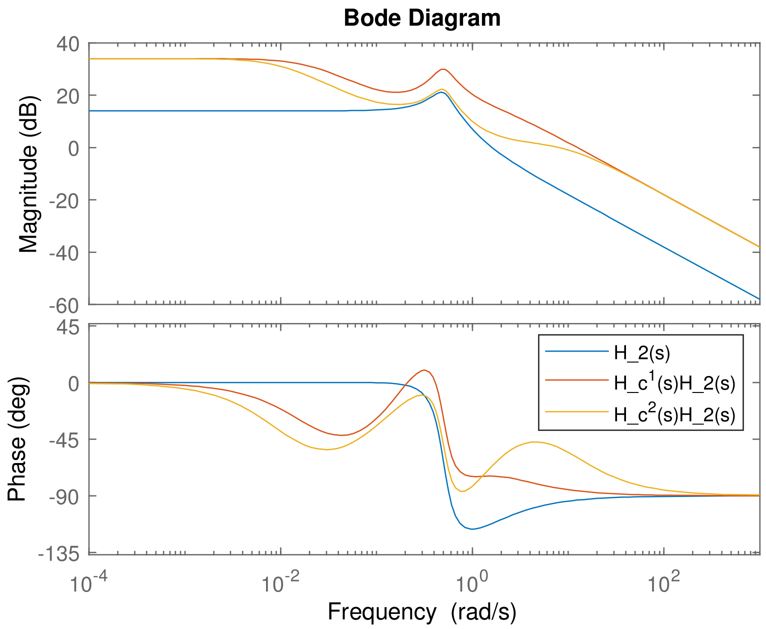

Figure 6.11: Lead-lag compensation for $H_2(s)$.

6.3 Response time

Given a second-order transfer function with one of the following structures:

$$H(s) = \frac{H(0)}{(1+s\tau_1)(1+s\tau_2)} \quad \text{(overdamped)} \tag{6.4}$$

$$H(s) = \frac{H(0)}{(1+s\tau)^2} \quad \text{(critically damped)} \tag{6.5}$$

$$H(s) = \frac{H(0)}{1+\frac{2\xi}{\omega_0}s+\frac{s^2}{\omega_0^2}} \quad \text{(underdamped)} \tag{6.6}$$

the response time T_r is defined as the time taken to reach 63% of the final value after a step change in the input. While for first-order systems, it corresponds to the time constant τ, for second-order system obtaining an exact expression is somewhat more complicate. For this reason, some approximate formulas are used instead, as follows:

$$T_r = \tau_1 + \tau_2 \quad \text{(overdamped)} \tag{6.7}$$

$$T_r = \tau + \tau \quad \text{(critically damped)} \tag{6.8}$$

$$T_r = \frac{1.5}{\omega_0} \quad \text{(underdamped)} \tag{6.9}$$

Question 6.8

Given the three following transfer functions:

$$H_1(s) = \frac{1}{(s+1)(2s+1)} \tag{6.10}$$

$$H_2(s) = \frac{1}{(s+1)^2} \tag{6.11}$$

$$H_3(s) = \frac{1}{(s^2+s+1)} \tag{6.12}$$

- Depending on the type of system, compute the response time T_r using the correct approximate formula among (6.7)-(6.9)
- Use the command `step` in MATLAB to read the real response time T_r for the three systems. To get enough resolution on the curve, add the time vector [0:0.01:10] to the end of the `step` command. For example, in the case of $H_1(s)$, you should write `step(1,[2 3 1],[0:0.01:10])`.

Solution: $H_1(s)$ is an overdamped system so $T_r \cong 1+2 = 3$ s. $H_2(s)$ is a critically damped system so $T_r \cong 1+1 = 2$ s. Finally, $H_3(s)$ is an underdamped system so $T_r \cong 1.5/1 = 1.5$ s.

Fig. 6.12 has been obtained using the following lines of MATLAB code:

```
>> H1 = tf([1],[2 3 1]);
>> H2 = tf([1],[1 2 1]);
>> H3 = tf([1],[1 1 1]);
>> step(H1,H2,H3,[0:0.01:10])
>> legend('H1(s)','H2(s)','H3(s)')
```

It shows that the response times are $T_r = 3.17$ s, $T_r = 2.14$ s and $T_r = 1.54$ s, respectively, which are quite close to the approximated calculated values.

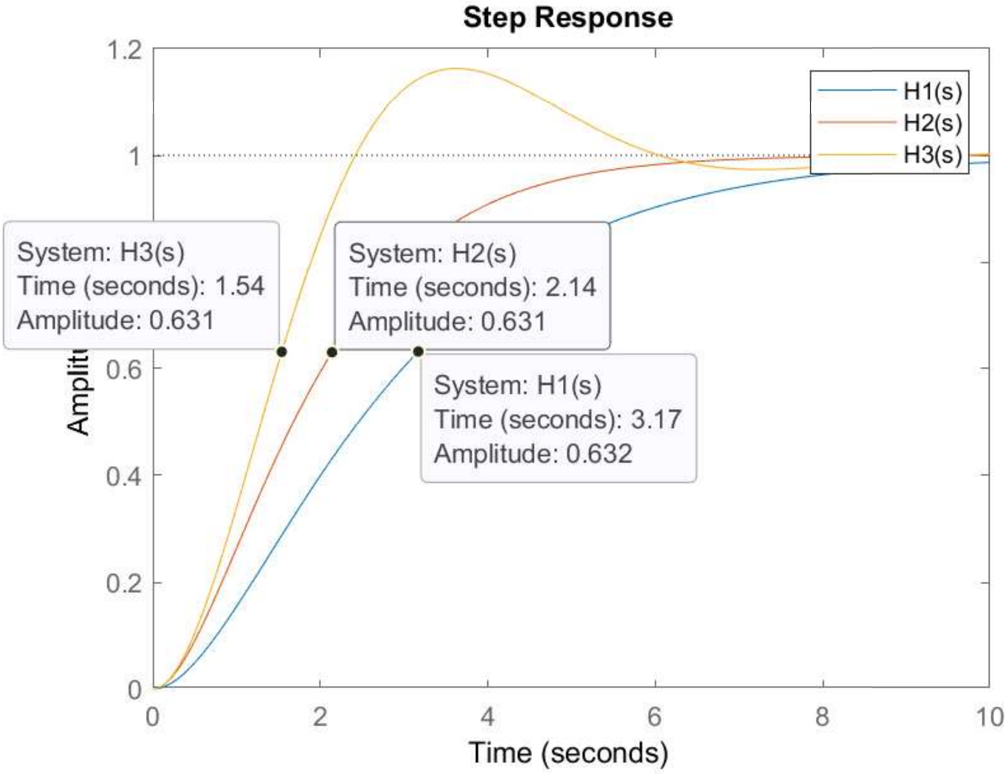

Figure 6.12: Response times.