8 Assignment

In this assignment, you will practice with the interpretation of the *sensitivity function* $S(s)$ and with the design of controllers that achieve some desired *tracking* and *disturbance rejection* performance.

8.1 Sensitivity function

We will use as a starting point the following LTI system:

$$Y(s) = H(s)U(s) + G(s)W(s) \tag{8.1}$$

where $U(s)$, $W(s)$ and $Y(s)$ denote the Laplace transforms of the control input, disturbance input and output signal, respectively, and the transfer functions $H(s)$ and $G(s)$ are given by:

$$H(s) = \frac{0.0025}{1 + 2000s} \qquad G(s) = \frac{1}{1 + 2000s} \tag{8.2}$$

You will use a PI controller:

$$C(s) = \frac{U(s)}{E(s)} = \frac{K_p(1 + sT_i)}{sT_i} \tag{8.3}$$

and a sensor with transfer function:

$$F(s) = \frac{Y_m(s)}{Y(s)} = \frac{1}{1 + 30s} \tag{8.4}$$

where $Y_m(s)$ denotes the measured output (the signal that can be compared with the reference to obtain the error signal: $E(s) = R(s) - Y_m(s)$).

In the Simulink model `vvb2.slx`, you will find the option to switch from automatic control (closed-loop) to manual control (open-loop). The file `vvb_data.m` contains the data for the process, as well as a shell for the rest of the tasks in the assignment. At first, we use the following values for the reference and the disturbance: $r(t) = 50$, $w(t) = 5$.

Question 8.1

What is the steady-state value of $u(t)$ that would ensure $y(\infty) = r(\infty) = 50$ in spite of the steady-state disturbance $w(\infty) = 5$? Set this value as the initial condition for the integrator in the PID-regulator block. In this way, you will avoid the transient, since you start the simulation working at the reference point. Insert this value also in `åpen sløyfe pådrag`, as it represents the input you apply when changing from automatic to manual control.

Solution. The steady-state value of $u(t)$ can be found from the equation:

$$y(\infty) = H(0)u(\infty) + G(0)w(\infty) \Rightarrow 50 = 0.0025u(\infty) + 1 \cdot 5 \Rightarrow u(\infty) = 18000$$

Question 8.2

Insert correct initial input and output values in the transfer functions $G(s)$, $H(s)$, $F(s)$. *The way you can check that the inserted values are correct is to simulate the response: if there is no transient, then the inserted values are correct.*

Solution: Since $w(t)$ contributes to the overall output with a value of 5, the initial output value in $G(s)$ must be 5, and since $G(0) = 1$, then also the initial input value must be set to 5. Then, the contribution of the process to the output will be $50 - 5 = 45$, which is the value to be set as initial output value in $H(s)$. Since $H(0) = 0.0025$, the initial input value in $H(s)$ should be set to 18000. Finally, since the sensor is measuring $y(t)$, its initial value should be set to 50, both in the input and in the output (since $F(0) = 1$).

The Skogestad's method for PID tuning finds controller parameters that provide a closed-loop transfer function:

$$H_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{1}{1 + s\tau_d} e^{-sT} \tag{8.5}$$

where τ_d is the desired closed-loop constant, and T denotes the transport delay. The controller parameters are chosen according to the following table, where $H(s)$ is the model of the plant to be controlled. k_1 is usually selected as $k_1 = 4$, but using a smaller value (e.g., $k_1 = 1.44$) gives improved disturbance compensation.

Question 8.3

Calculate k_p and T_i using Skogestad's method, and insert the computed values into `vvb_data.m`. Specify τ_d as half of the open-loop time constant of $H(s)$, and use $k_1 = 1.44$ for improved disturbance compensation.

$H(s)$	k_p	T_i	T_d
$\frac{H(0)}{s}e^{-sT}$	$\frac{1}{H(0)(\tau_d+T)}$	$k_1(\tau_d+T)$	0
$\frac{H(0)}{1+s\tau}e^{-sT}$	$\frac{\tau}{H(0)(\tau_d+T)}$	$\min[\tau, k_1(\tau_d+T)]$	0
$\frac{H(0)}{s(1+s\tau)}e^{-sT}$	$\frac{1}{H(0)(\tau_d+T)}$	$k_1(\tau_d+T)$	τ
$\frac{H(0)e^{-sT}}{(1+s\tau_1)(1+s\tau_2)}$ $(\tau_1 > \tau_2)$	$\frac{\tau_1}{H(0)(\tau_d+T)}$	$\min[\tau_1, k_1(\tau_d+T)]$	τ_2
$\frac{H(0)}{s^2}e^{-sT}$	$\frac{1}{4H(0)(\tau_d+T)^2}$	$4(\tau_d+T)$	$4(\tau_d+T)$

Table 8.1: Skogestad tuning table.

Solution: We calculate:

$$k_p = \frac{\tau}{H(0)(\tau_d+T)} = \frac{2000}{0.0025(1000+0)} = 800$$
$$T_i = \min[\tau, k_1(\tau_d+T)] = \min(2000, 1.44(1000+0)) = 1440$$

Question 8.4

Find by hand the expression for the open-loop transfer function $H_{ol}(s) = Y_m(s)/E(s)$ (assuming $W(s) = 0$). Then, implement in `vvb_data.m` the expressions for $H(s)$, $C(s)$, $F(s)$, and calculate therein the transfer function $H_{ol}(s)$. Does the transfer function computed in MATLAB match the one that you computed? *Hint: Since the m-file is not complete yet, you can select the part of the file you want to run, right-click and select Evaluate selection.*

Solution: The expression for the open-loop transfer function can be obtained as follows:

$$Y_m(s) = F(s)Y(s) = F(s)H(s)U(s) + F(s)G(s)W(s) = F(s)H(s)C(s)E(s) + F(s)G(s)W(s)$$

By assuming that $W(s) = 0$, we get:

$$Y_m(s) = F(s)H(s)C(s)E(s) \quad \Rightarrow \quad H_{ol}(s) = F(s)H(s)C(s)$$

so that:

$$H_{ol}(s) = \frac{1}{1+30s} \cdot \frac{0.0025}{1+2000s} \cdot \frac{800(1+1440s)}{1440s} = \frac{2(1+1440s)}{8.64 \cdot 10^7 s^3 + 2.92 \cdot 10^6 s^2 + 1440s}$$

which matches the transfer function computed by MATLAB:

```
H_ol =

2880 s + 2
-----------------------------------
8.64e07 s^3 + 2.923e06 s^2 + 1440 s

Continuous-time transfer function.
```

Question 8.5

Find by hand the expressions for the sensitivity function $S(s) = E(s)/R(s)$ and for the complementary sensitivity function $T(s) = Y_m(s)/R(s)$ (assuming $W(s) = 0$). Calculate their values and compare them with those returned by MATLAB (the script includes the function `minreal` to perform a pole/zero cancellation and reduce the order of the transfer function computed by MATLAB).

Solution: The expression for the sensitivity function can be obtained taking into account that:

$$E(s) = R(s) - Y_m(s) = R(s) - F(s)Y(s) = R(s) - F(s)H(s)U(s) = R(s) - F(s)H(s)C(s)E(s)$$

which leads to:

$$(1 + F(s)H(s)C(s))\, E(s) = R(s) \quad \Rightarrow \quad S(s) = \frac{1}{1 + F(s)H(s)C(s)}$$

so:

$$S(s) = \frac{1}{1 + \frac{2(1+1440s)}{8.64\cdot 10^7 s^3 + 2.92\cdot 10^6 s^2 + 1440s}} = \frac{8.64 \cdot 10^7 s^3 + 2.9 \cdot 10^6 s^2 + 1440s}{8.64 \cdot 10^7 s^3 + 2.93 \cdot 10^6 s^2 + 4320s + 2}$$

as confirmed by MATLAB:

```
S =

8.64e07 s^3 + 2.923e06 s^2 + 1440 s
---------------------------------------
8.64e07 s^3 + 2.923e06 s^2 + 4320 s + 2
```

On the other hand, the expression for the complementary sensitivity function is obtained by considering:

$$Y_m(s) = F(s)Y(s) = F(s)H(s)U(s) = F(s)H(s)C(s)\,(R(s) - Y_m(s))$$

which leads to:

$$(1 + F(s)H(s)C(s))\, Y_m(s) = F(s)H(s)C(s)R(s) \quad \Rightarrow \quad T(s) = \frac{F(s)H(s)C(s)}{1 + F(s)H(s)C(s)}$$

so:

$$T(s) = \frac{\frac{2(1+1440s)}{8.64\cdot 10^7 s^3 + 2.92\cdot 10^6 s^2 + 1440s}}{1 + \frac{2(1+1440s)}{8.64\cdot 10^7 s^3 + 2.92\cdot 10^6 s^2 + 1440s}} = \frac{1 + 1440s}{4.32 \cdot 10^7 s^3 + 1.46 \cdot 10^6 s^2 + 2160s + 1}$$

which corresponds to the function returned by MATLAB (although obtained in a different form):

```
T =

3.333e-05 s + 2.315e-08
---------------------------------------
s^3 + 0.03383 s^2 + 5e-05 s + 2.315e-08
```

To understand the meaning of the sensitivity function $S(s)$, we will perform some simulations.

Question 8.6

Run the completed file `vvb_data.m` and read the values of $|S(j\omega)|$ from the Bode plots for three different frequencies: $\omega_1 = 0.0001$ rad/s, $\omega_2 = 0.001$ rad/s and $\omega_3 = 0.01$ rad/s. Then, change the reference signal to a sine wave and simulate the feedback control system with the reference signal $r(t) = 50 + \sin(\omega t)$ with frequencies ω_1, ω_2 and ω_3, respectively, **while using a constant disturbance signal $w = 5$. Also, change the time at which you switch from automatic to manual control so that it matches the total length of the simulation**. Check the obtained results and compare the error signal with the value of $|S(j\omega)|$ that you computed.

Solution: The Bode diagrams of $S(j\omega)$, $T(j\omega)$ and $H_{ol}(j\omega)$ are shown in Fig. 8.1. It can be seen that $|H(0.0001j)| = 10^{-22.8/20} = 0.07$, $|H(0.001j)| = 10^{-2.58/20} = 0.74$ and $|H(0.01j)| = 10^{-0.22/20} = 0.97$. The obtained results are shown in Figs. 8.2-8.4: it can be seen that the sensitivity function $S(s)$ describes the magnitude of tracking error that will be obtained when using a sinusoidal reference signal (note that $S(0) = 0$ which is the reason why the constant part of the reference signal does not provide contribution to the steady-state error.

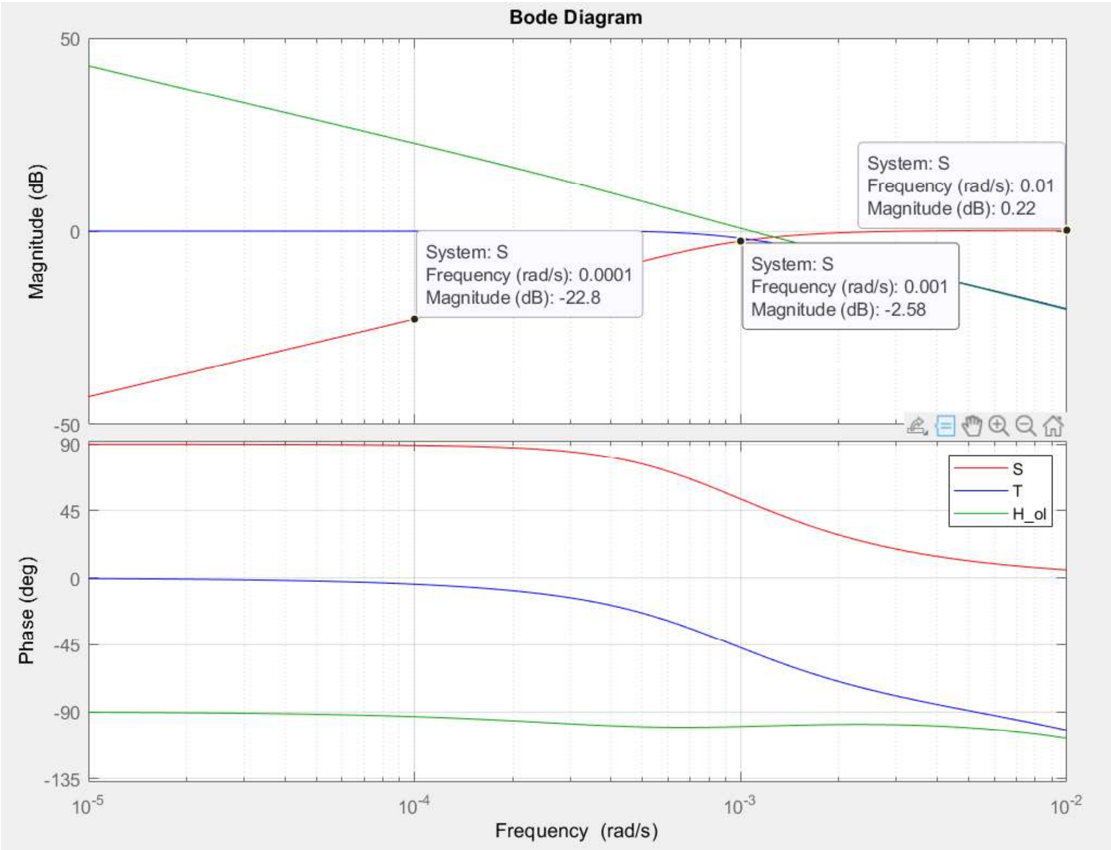

Figure 8.1: Bode diagrams of $S(j\omega)$, $T(j\omega)$ and $H_{ol}(j\omega)$.

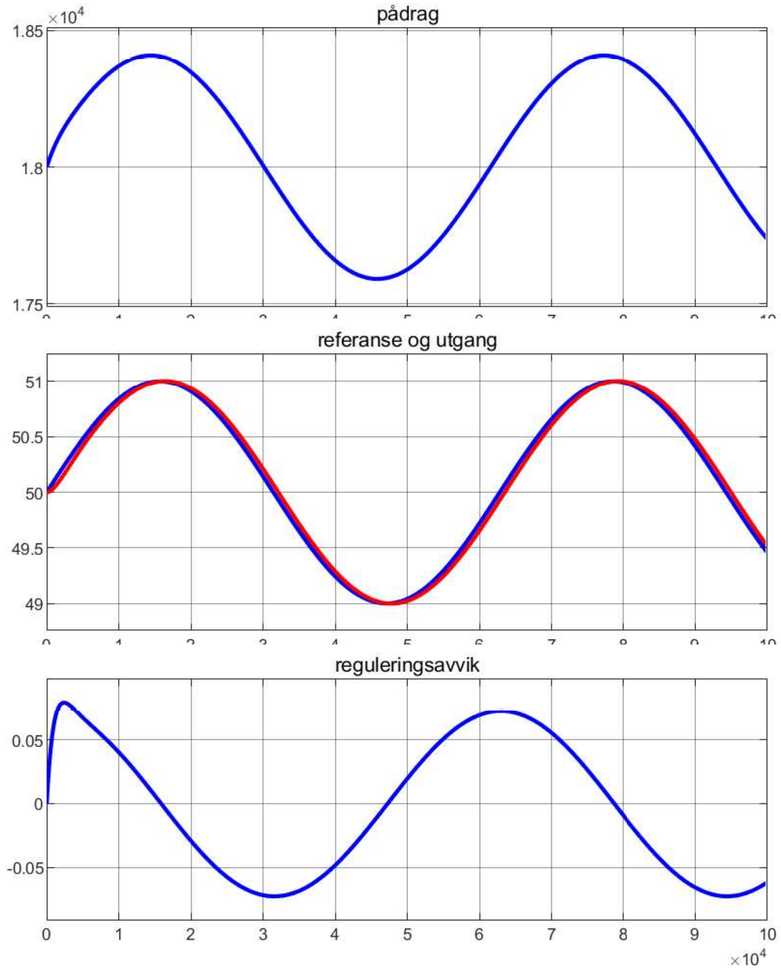

Figure 8.2: Simulation with the reference $r(t) = 50 + \sin(0.0001t)$ and the disturbance $w(t) = 5$.

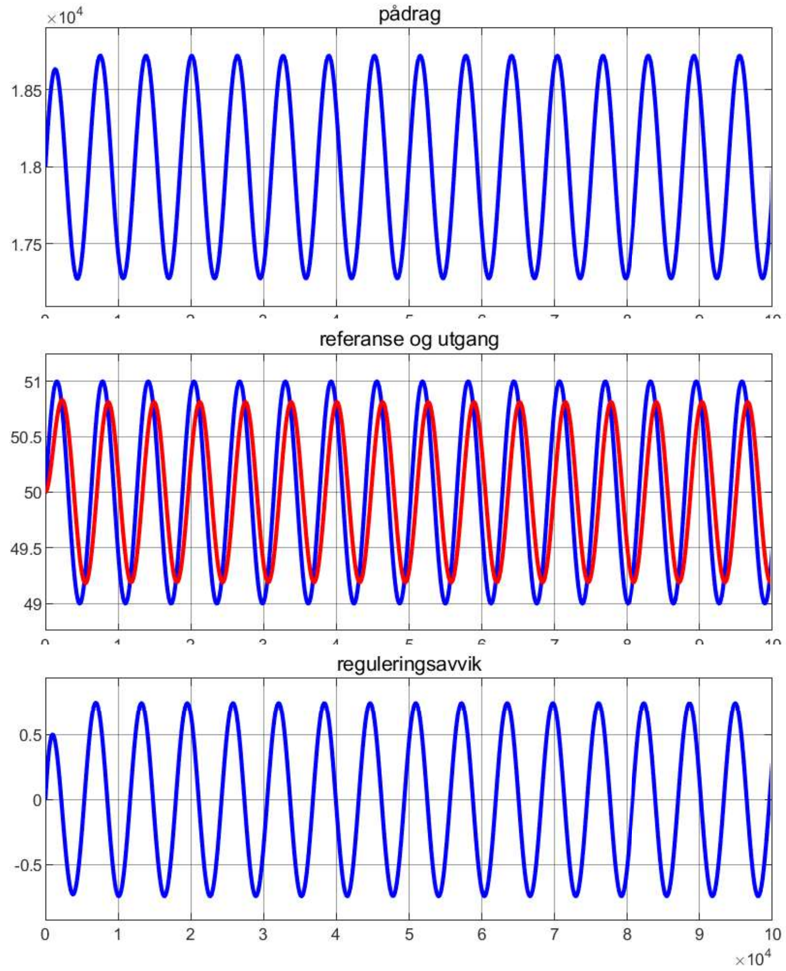

Figure 8.3: Simulation with the reference $r(t) = 50 + \sin(0.001t)$ and the disturbance $w(t) = 5$.

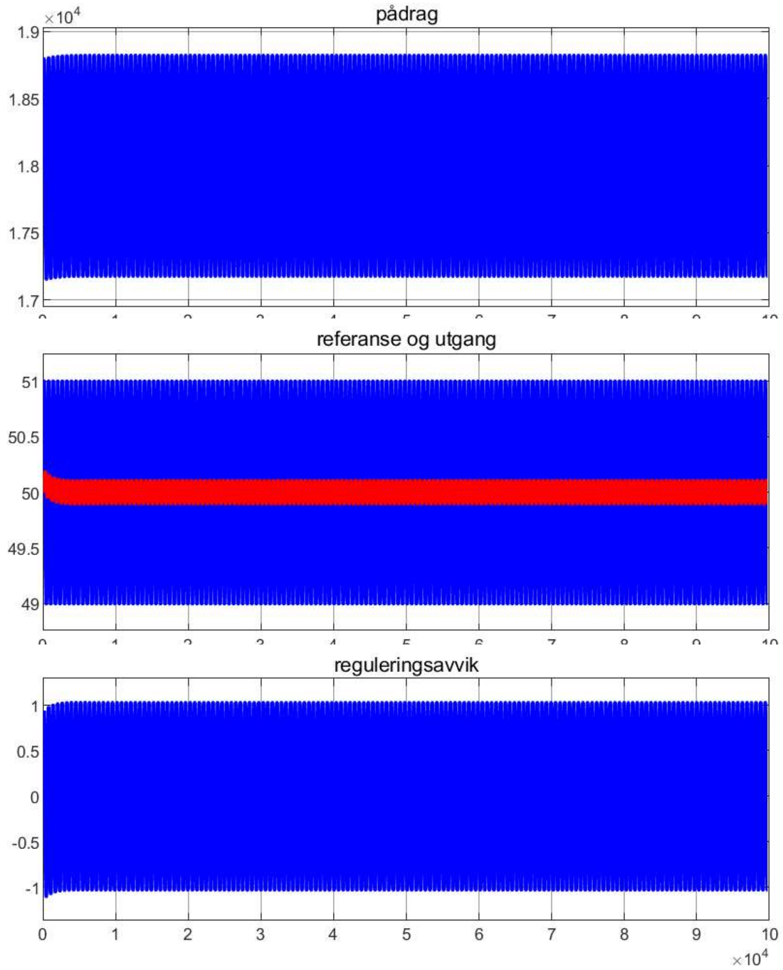

Figure 8.4: Simulation with the reference $r(t) = 50 + \sin(0.01t)$ and the disturbance $w(t) = 5$.

Question 8.7

We will now use a constant reference signal $r(t) = 50$, but we will consider a sinusoidal disturbance $w(t) = 5+\sin(1.37\cdot 10^{-4}t)$. Note that the frequency $\omega = 1.37\cdot 10^{-4}$ corresponds to $|S(j\omega)|_{dB} = -20$. Choose the time at which you switch from automatic (closed-loop) to manual (open-loop) control as half the length of the simulation. Observe the behavior of the error signal and discuss about it in relationship with the value of the sensitivity function.

Solution: The obtained simulation is shown in Fig. 8.5. We can see that another interpretation of the sensitivity function is that it describes how much the disturbance rejection is improved when closing the loop:

$$S(s) = \frac{E_{cl}(s)}{E_{ol}(s)}$$

In particular, since $|S(1.37 \cdot 10^{-4} j)| = 10^{-20/20} = 0.1$, the disturbance rejection is 10 times more effective in closed-loop than in open-loop.

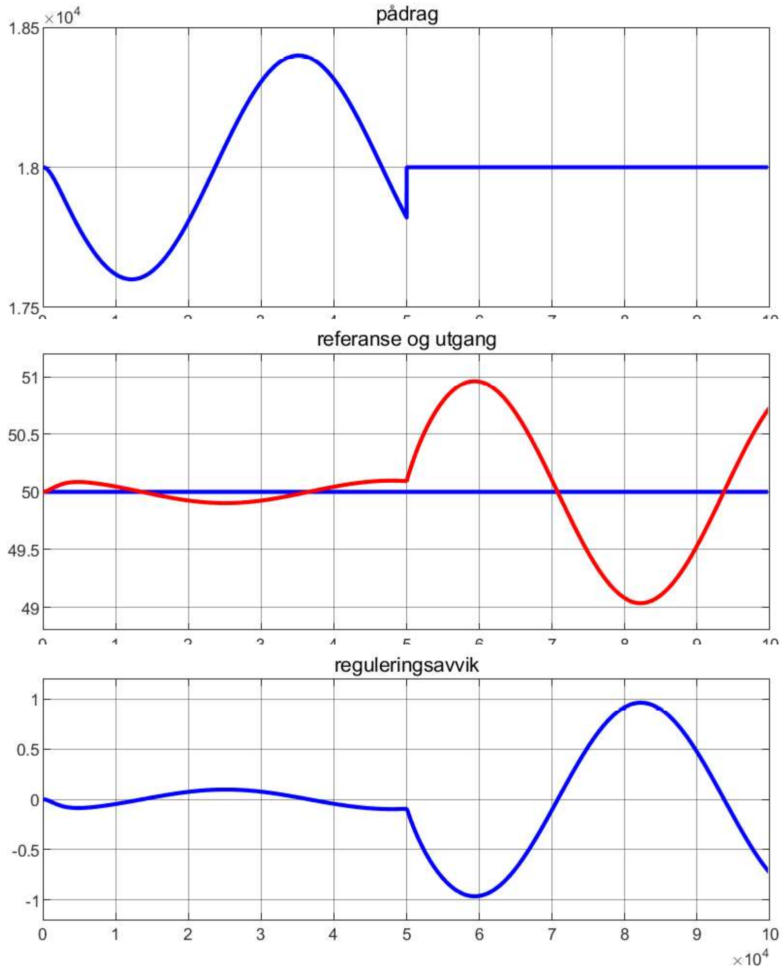

Figure 8.5: Simulation with reference $r(t) = 50$ and disturbance $w(t) = 5 + \sin(1.37 \cdot 10^{-4} t)$.

8.2 Design of the controller transfer function

We will now consider the design of a controller $C(s)$ for the following plant:

$$P(s) = \frac{1}{s+1} \tag{8.6}$$

with two structures of the controller transfer function:

$$C_0(s) = \frac{k}{D(s)} \tag{8.7}$$

$$C_2(s) = \frac{as^2 + bs + c}{D(s)} \tag{8.8}$$

where a, b, c (or k) are parameters to be designed, and $D(s)$ denotes a polynomial to be designed.

We would like to achieve the following requirements on tracking and disturbance rejection performance:

- finite steady-state error $e_\infty = 0.5$ when tracking the reference signal: $r(t) = 5t \cdot 1(t)$
- zero contribution to the steady-state error e_∞ when considering a sinusoidal disturbance $w(t) = A\sin(t)$, with $A \in \mathbb{R}$

Question 8.8

What is the minimum-order polynomial $D(s)$ that allows achieving the desired tracking/disturbance rejection performance?

Solution: According to the internal model principle, in order to get a zero steady-state error on a ramp reference signal, the direct loop transfer function $P(s)C(s)$ should contain two poles in the origin. A non-zero (but finite) steady-state error would be obtained by having one less pole in $s = 0$, which means that $P(s)C(s)$ should have a pole in the origin. Since $P(s)$ does not have a pole in the origin, we need to include it in the controller transfer function, so $D(s)$ should have a root $s = 0$. On the other hand, to get a zero contribution to e_∞ when $P(s)$ is affected by sinusoidal disturbances, we need to add a term $s^2 + \omega^2$, where ω is the frequency of the disturbance, in the denominator of the controller. Taking into account this discussion, $D(s)$ should be chosen as follows:

$$D(s) = s(s^2 + 1)$$

Higher-order polynomials could be used (as long as they contain poles in 0, $-j$ and j), but this is the minimum-order solution.

Question 8.9

By using the Routh-Hurwitz criterion, show that using $C_0(s)$ with the choice of $D(s)$ from Question 8.8 would lead to closed-loop instability independently from the choice of $k \in \mathbb{R}$. *Hint: When applying the Routh-Hurwitz criterion you will find that one of the elements in the first column of the table becomes* 0. *In order to proceed further, you can replace* 0 *with an infinitesimally small parameter* $\epsilon > 0$*, and check that the inequalities obtained when completing the table are incompatible with each other.*

Solution: The closed-loop transfer functions with the controller $C_0(s)$ are given by:

$$H_{YR}(s) = \frac{P(s)C_0(s)}{1+P(s)C_0(s)} = \frac{\frac{1}{s+1}\frac{k}{s(s^2+1)}}{1+\frac{1}{s+1}\frac{k}{s(s^2+1)}} = \frac{k}{s(s+1)(s^2+1)+k} = \frac{k}{s^4+s^3+s^2+s+k}$$

$$H_{ER}(s) = \frac{1}{1+P(s)C_0(s)} = \frac{1}{1+\frac{1}{s+1}\frac{k}{s(s^2+1)}} = \frac{s(s+1)(s^2+1)}{s(s+1)(s^2+1)+k} = \frac{s(s+1)(s^2+1)}{s^4+s^3+s^2+s+k}$$

so that the Routh-Hurwitz criterion should be used to check stability of the polynomial $s^4 + s^3 + s^2 + s + k$. The Routh-Hurwitz table is built as follows: from which we get two inequalities:

1	1	k
1	1	0
ϵ	k	0
$1-\frac{k}{\epsilon}$	0	0
k	0	0

Table 8.2: Routh-Hurwitz table for the polynomial $s^4 + s^3 + s^2 + s + k$ (with ϵ instead of 0).

$$\begin{cases} 1-\frac{k}{\epsilon} > 0 \\ k > 0 \end{cases}$$

but the first inequality corresponds to $\frac{k}{\epsilon} < 1 \Rightarrow k < \epsilon$ which is incompatible with $k > 0$ (since ϵ is infinitesimally small).

Question 8.10

By using the Routh-Hurwitz criterion, find the inequalities that must be satisfied by the design parameters a, b, c so that closed-loop BIBO stability is obtained.

Solution: The closed-loop transfer functions with the controller $C_2(s)$ are given by:

$$H_{YR}(s) = \frac{P(s)C_2(s)}{1+P(s)C_2(s)} = \frac{\frac{1}{s+1}\frac{as^2+bs+c}{s(s^2+1)}}{1+\frac{1}{s+1}\frac{as^2+bs+c}{s(s^2+1)}} = \frac{as^2+bs+c}{s^4+s^3+(a+1)s^2+(b+1)s+c}$$

$$H_{ER}(s) = \frac{1}{1+P(s)C_2(s)} = \frac{1}{1+\frac{1}{s+1}\frac{as^2+bs+c}{s(s^2+1)}} = \frac{s(s+1)(s^2+1)}{s^4+s^3+(a+1)s^2+(b+1)s+c}$$

so that the Routh-Hurwitz criterion should be used to check stability of the polynomial $s^4 + s^3 + (a+1)s^2 + (b+1)s + c$. The Routh-Hurwitz table is built as follows:

which leads to inequalities:

$$\begin{cases} a-b > 0 \\ b+1-\frac{c}{a-b} > 0 \\ c > 0 \end{cases}$$

1	$a+1$	c
1	$b+1$	0
$a-b$	c	0
$b+1-\frac{c}{a-b}$	0	0
c	0	0

Table 8.3: Routh-Hurwitz table for the polynomial $s^4+s^3+(a+1)s^2+(b+1)s+c$.

Question 8.11

By using the final value theorem, determine the value of c that leads to finite steady-state error $e_\infty = 0.5$ when tracking the reference signal: $r(t) = 5t \cdot 1(t)$. Is this value compatible with the closed-loop BIBO stability requirement?

Solution: The Laplace transform of $r(t)$ is:

$$R(s) = \frac{5}{s^2}$$

so we have:

$$e_\infty = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sH_{ER}(s)R(s) = \lim_{s\to 0} s\frac{s(s+1)(s^2+1)}{s^4+s^3+(a+1)s^2+(b+1)s+c}\frac{5}{s^2} = \frac{5}{c}$$

which leads to:

$$\frac{5}{c} = 0.5 \quad \Longrightarrow \quad c = \frac{5}{0.5} = 10$$

This value is compatible with the closed-loop stability requirement, since $c > 0$.

Question 8.12

Use $D_0(s)$ from Question 8.8, c from Question 8.11, and choose parameters a, b, c so that the inequalities obtained in Question 8.10 are satisfied. Create a Simulink scheme that simulates the plant $P(s)$ in a feedback control scheme with the designed controller $C_2(s)$. Show that the tracking/disturbance requirements are satisfied when $r(t) = 5t \cdot 1(t)$ and $w(t) = A\sin(t)$ (choose $A \neq 0$ as you wish). Choose a length of the simulation that allows checking that $e_\infty = 0.5$. Attach the Simulink scheme and the plots that you get as part of the solution.

Solution: Fig. 8.6 shows the Simulink scheme with $a = 6$ and $b = 3$. The results shown in Figs. 8.7-8.8 show that the sinusoidal disturbance is rejected at steady-state, and the final error on tracking the ramp signal is $e_\infty = 0.5$.

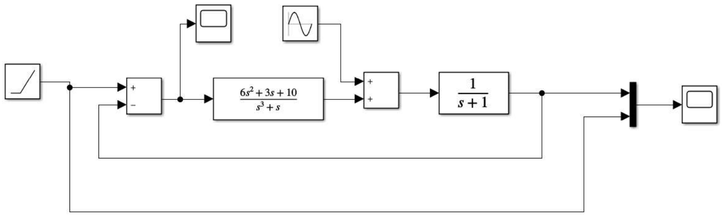

Figure 8.6: Simulink scheme.

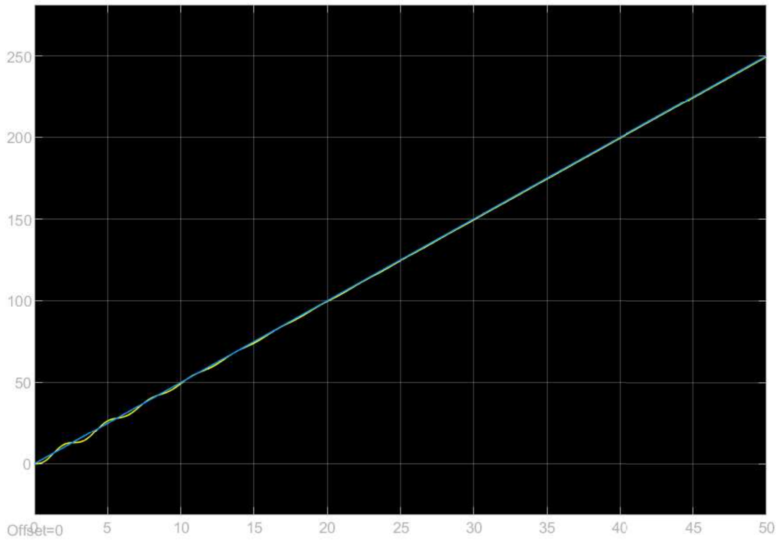

Figure 8.7: Signals $r(t)$ (blue) and $y(t)$ (yellow).

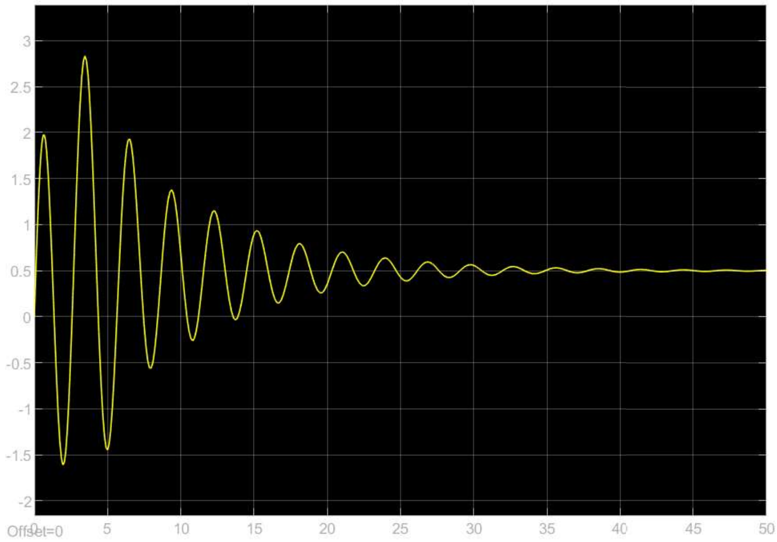

Figure 8.8: Error signal $e(t)$.

Question 8.13

Use the function bode to plot the Bode diagram of the sensitivity function $S(s)$. Provide an interpretation of the magnitude plot in connection with the disturbance rejection requirement.

Solution: The sensitivity function can be obtained in MATLAB with the following commands:

```
>> P = tf([1],[1 1])

P =

1
-----
s + 1

Continuous-time transfer function.

>> C = tf([6 3 10],[1 0 1 0])

C =

6 s^2 + 3 s + 10
----------------
s^3 + s
```

```
Continuous-time transfer function.

>> HER = 1/(1+P*C)

HER =

s^4 + s^3 + s^2 + s
----------------------------
s^4 + s^3 + 7 s^2 + 4 s + 10

Continuous-time transfer function.

>> bode(HER)
```

The resulting figure from the bode command is shown in Fig. 8.9. It can be seen that the magnitude $|S(j\omega)|$ goes to $-\infty$ dB (which corresponds to 0) when $\omega = 1$ rad/s. Since the sensitivity function describes how much the disturbance rejection is improved by closing the loop with respect to the open loop case, we can say that *the rejection of a sinusoidal disturbance at frequency* 1 rad/s *is improved infinitely when closing the loop*. By the way, we also see that there are some frequencies (in particular the interval $\omega \in [1.5, 3.5]$ rad/s) where the closed-loop system will be more affected than the open-loop system by sinusoidal disturbances.

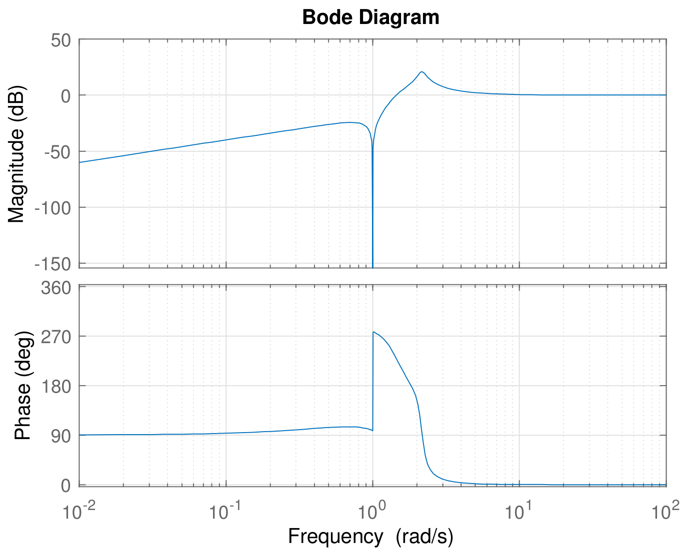

Figure 8.9: Bode diagram of the sensitivity function $S(s)$.