THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM I: MAT300 Vector Analysis
DATE: 16. December 2021, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators permitted in accordance with TN faculty rules
THE EXERCISE SHEET CONSISTS OF 4 EXERCISES ON 3 PAGES
+ 1 PAGE WITH FORMULAS.
EACH OF THE 10 PARTS 1a, 1b, 2a, 2b, 3a, 3b, 3c, 3d, 4a, 4b ARE WORTH EQUAL MARKS.

EXERCISE 1

Consider a piece of wire given by the curve $\mathscr{C}_1$: $\mathbf{r}_1(t) = t\cos t\,\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\,\mathbf{j} - t\sin t\,\mathbf{k}$, $0 \leq t \leq \pi$.

a) (i) Find a tangent vector to $\mathscr{C}_1$ at the point corresponding to $t = \frac{\pi}{2}$. Note that it is NOT necessary for the tangent vector to have unit length.

(ii) Find the mass of the wire knowing that the density at each point of $\mathscr{C}_1$ is given by the function

$$\rho(x, y, z) = \frac{x^2 + z^2}{1 + (\frac{3\sqrt{2}}{4}y)^{\frac{2}{3}}}.$$

Let the curve $\mathscr{C}_2 \subset \mathbb{R}^3$ be the part of the intersection of the surfaces $2x - y = 3$ and $z = x^2 + y^2$ from $(1, -1, 2)$ to $(2, 1, 5)$. Consider the vector field

$$\mathbf{F}(x, y, z) = zy\,\mathbf{i} + (ye^{y^2} + xz)\,\mathbf{j} + xy\,\mathbf{k}\,.$$

b) (i) Parametrise the curve $\mathscr{C}_2$ by using $y = t$ as a parameter.

(ii) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.

(iii) Compute the line integral

$$\int_{\mathscr{C}_2} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 2

Let $D \subset \mathbb{R}^2$ be the domain described by $\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$. Notice that D is treated as a y-simple domain.

a) (i) Sketch the given domain D in the xy-plane. Make sure to clearly label all lines and curves that you draw, and clearly show the coordinates of each vertex of D.

(ii) Describe D as an x-simple domain.

(iii) Compute the double integral

$$\iint_D \cos(\frac{\pi y^3}{2})\, dx\, dy\,.$$

Let $\mathcal{P} \subset \mathbb{R}^3$ be the plane that passes through the point $(3, -2, 0)$ and is perpendicular to the line with direction vector $\mathbf{v} = (0, -1, 2)$, and let $\mathscr{S}$ be the surface in $\mathcal{P}$ whose projection onto the xy-plane is D.

b) (i) Find the equation of the plane $\mathcal{P}$.

(ii) Compute the area of the surface $\mathscr{S}$.

EXERCISE 3

Let $T \subset \mathbb{R}^3$ be the solid region above the xy-plane that lies between the cone $x^2+y^2-z^2 = 0$ and the cone $3(x^2+y^2)-z^2 = 0$, and is inside the sphere $x^2+y^2+z^2 = 9$.

a) (i) Suppose that a function $f(x, y, z)$ satisfies

$$f(-x, y, z) = -f(x, y, z) \quad \text{for all points} \quad (x, y, z) \in \mathbb{R}^3\,.$$

Carefully explain why it then follows that

$$\iiint_T f(x, y, z)\, dV = 0\,.$$

(ii) Find the angle ϕ formed by the line $z = \sqrt{3}y$ (in the yz-plane) and the z-axis.

b) Compute the integral

$$\iiint_T (3z - 4xy)\, dV\,.$$

Now, let $\mathscr{S}$ be the part of the inverted paraboloid $z = 4 - x^2 - y^2$ in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ that is inside the cylinder $x^2 + y^2 = 1$. Let $\mathbf{F}$ be the vector field given by the formula

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} - z\,\mathbf{k}\,.$$

c) Compute the flux of $\mathbf{F}$ downward through the surface $\mathscr{S}$.

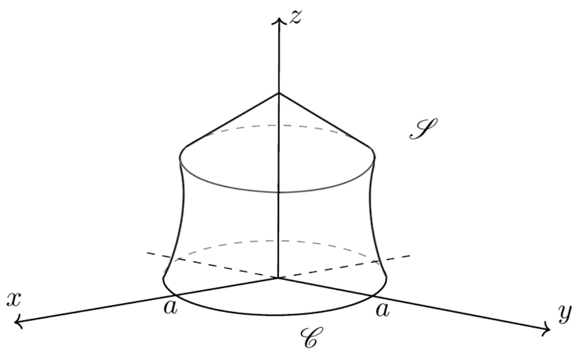
d) Let $f(x, y) = 4 - x^2 - y^2$. Give a unit vector in the direction of the maximum rate of increase of f at the point $P = (-1, -1)$.

EXERCISE 4

Consider the vector field $\mathbf{F}(x, y, z) = (xz - \dfrac{y^3 \cos z}{3})\,\mathbf{i} + \dfrac{x^3 \cos z}{3}\,\mathbf{j} + (xy + z)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Consider the following surface $\mathscr{S} \subset \mathbb{R}^3$:



whose boundary $\mathscr{C}$ is the circle in the xy-plane of radius a and centred at the origin. Note that $\mathscr{S}$ is open on the bottom. We equip $\mathscr{S}$ with the **inwards** pointing unit normal vector field $\hat{\mathbf{N}}_1$.

b) (i) Indicate (in a drawing) the induced orientation on the boundary circle $\mathscr{C}$. Explain the method you used.

(ii) Let D be the disc on the xy-plane that is bounded by the circle $\mathscr{C}$, equipped with the unit normal vector field $\hat{\mathbf{N}}_2$. Indicate in a drawing which direction $\hat{\mathbf{N}}_2$ should be pointing so that Stokes' theorem gives the following equality:

$$\iint_{\mathscr{S}} (\nabla \times \mathbf{F}) \bullet \hat{\mathbf{N}}_1 \, dS = \iint_{D} (\nabla \times \mathbf{F}) \bullet \hat{\mathbf{N}}_2 \, dS\,.$$

(iii) Let $\mathscr{C}$ have the clockwise orientation, when viewed from above. Compute the circulation of $\mathbf{F}$ around $\mathscr{C}$, given by

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \,du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \,dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S}$ parametrised by $\mathbf{r}(x,y)$, $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f \left|\frac{\partial\mathbf{r}}{\partial x} \times \frac{\partial\mathbf{r}}{\partial y}\right| \,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}\,:\,G(x,y,z) = c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}\,:\,G(x,y,z) = c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint_D \nabla\bullet\mathbf{F}\ dV = \oiint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\ dS = \oint_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$, $\sin^4\theta = \frac{1}{4}(1-\cos(2\theta))^2$, $\cos^4\theta = \frac{1}{4}(\cos(2\theta)+1)^2$.