Solutions

Exercise 1:

a) i) A tangent vector on $\mathcal{C}_1$ at the point $\underset{\sim}{r}_1(\frac{\pi}{2})$ is given by $\frac{d\underset{\sim}{r}_1(t)}{dt}$ evaluated at $t=\frac{\pi}{2}$:

$$\frac{d\underset{\sim}{r}_1(t)}{dt} = (\cos t - t\sin t)\underset{\sim}{i} + \cancel{\frac{2\sqrt{2}}{3}}\cdot\cancel{\frac{3}{2}}\cdot\sqrt{t}\,\underset{\sim}{j} - (\sin t + t\cos t)\underset{\sim}{k}.$$

so, our tangent vector is $\boxed{\frac{d\underset{\sim}{r}_1}{dt}(\frac{\pi}{2}) = \left(-\frac{\pi}{2}, \sqrt{\pi}, -1\right).}$

ii) The mass of the wire $\mathcal{C}_1$ is given by the line integral:

$$\int_{\mathcal{C}_1} \frac{x^2+z^2}{1+\left(\frac{3\sqrt{2}}{4}y\right)^{2/3}}\,ds,$$

which is equal to

$$\int_0^{\pi} \frac{t^2\cos^2 t + t^2\sin^2 t}{1+\left(\cancel{\frac{3\sqrt{2}}{4}}\cdot\cancel{\frac{2\sqrt{2}}{3}}\,t^{\frac{3}{2}}\right)^{\frac{2}{3}}}\left|\frac{d\underset{\sim}{r}_1(t)}{dt}\right|dt$$

$$= \boxed{\int_0^{\pi} \frac{t^2}{1+t}\left|\frac{d\underset{\sim}{r}_1(t)}{dt}\right|dt.}$$

we have $\left|\frac{d\underset{\sim}{r}_1(t)}{dt}\right| = \sqrt{(\cos t - t\sin t)^2 + (\sqrt{2}\sqrt{t})^2 + (-(\sin t + t\cos t))^2}$

$$= \sqrt{t^2\cos^2 t + \cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + 2t + \sin^2 t + \cancel{2t\cos t\sin t}}$$

$$= \sqrt{1+t^2+2t} = \sqrt{(1+t)^2} = \boxed{1+t}$$

So we get $\displaystyle\int_0^{\pi} \frac{t^2}{\cancel{1+t}} \cancel{(1+t)}\, dt = \int_0^{\pi} t^2 dt$

$$= \frac{1}{3}\left[t^3\right]_0^{\pi} = \boxed{\frac{\pi^3}{3}}.$$

b) i) We have $y = \boxed{t}$, so $2x - y = 3 \Rightarrow 2x - t = 3$

$$\Rightarrow x = \boxed{\frac{3+t}{2}}$$

also, we have $z = x^2 + y^2 \Rightarrow z = \dfrac{(3+t)^2}{4} + t^2$

$$\Rightarrow z = \boxed{\frac{9}{4} + \frac{3t}{2} + \frac{5}{4}t^2}$$

So, C_2 is parametrised by

$$\underset{\sim}{r}_2(t) = \left(\frac{3+t}{2}\right)\underset{\sim}{i} + t\,\underset{\sim}{j} + \left(\frac{5}{4}t^2 + \frac{3}{2}t + \frac{9}{4}\right)\underset{\sim}{k}, \quad -1 \leqslant t \leqslant 1.$$

b) ii) Suppose there is a scalar potential ϕ such that

$\underset{\sim}{\nabla}\phi = \underset{\sim}{F} \Rightarrow$ then:
$$\begin{cases} (1) & \dfrac{\partial \phi}{\partial x} = zy \\ (2) & \dfrac{\partial \phi}{\partial y} = ye^{y^2} + xz \\ (3) & \dfrac{\partial \phi}{\partial z} = xy \end{cases}$$

$(1) \Rightarrow \phi(x,y,z) = \int zy\,dx \Rightarrow \phi(x,y,z) = xzy + c_1(y,z)$

$$\textcircled{2} \Rightarrow \frac{\partial\left(xzy + C_1(y,z)\right)}{\partial y} = ye^{y^2} + xz$$

$$\Rightarrow \frac{\partial C_1(y,z)}{\partial y} + \cancel{xz} = ye^{y^2} + \cancel{xz}$$

$$\Rightarrow \frac{\partial C_1(y,z)}{\partial y} = ye^{y^2} \Rightarrow C_1(y,z) = \frac{e^{y^2}}{2} + C_2(z)$$

$$\textcircled{3} \Rightarrow \frac{\partial\left(xyz + \frac{e^{y^2}}{2} + C_2(z)\right)}{\partial z} = xy$$

$$\Rightarrow \cancel{xy} + 0 + \frac{\partial C_2(z)}{\partial z} = \cancel{xy}$$

$$\Rightarrow \frac{\partial C_2(z)}{\partial z} = 0 \Rightarrow C_2(z) = C_3$$

constant.

Hence, $\boxed{\phi(x,y,z) = xyz + \frac{e^{y^2}}{2} + C_3}$

Checking that indeed $\underset{\sim}{\nabla}\phi = \underset{\sim}{F}$: $\underset{\sim}{\nabla}\phi = yz\,\underset{\sim}{i} + \left(xz + \frac{\cancel{2}y}{\cancel{2}}e^{y^2}\right)\underset{\sim}{j} + xy\,\underset{\sim}{k}$

$= \underset{\sim}{F}$ ✓

b) iii) Since $\underset{\sim}{F}$ is conservative, then

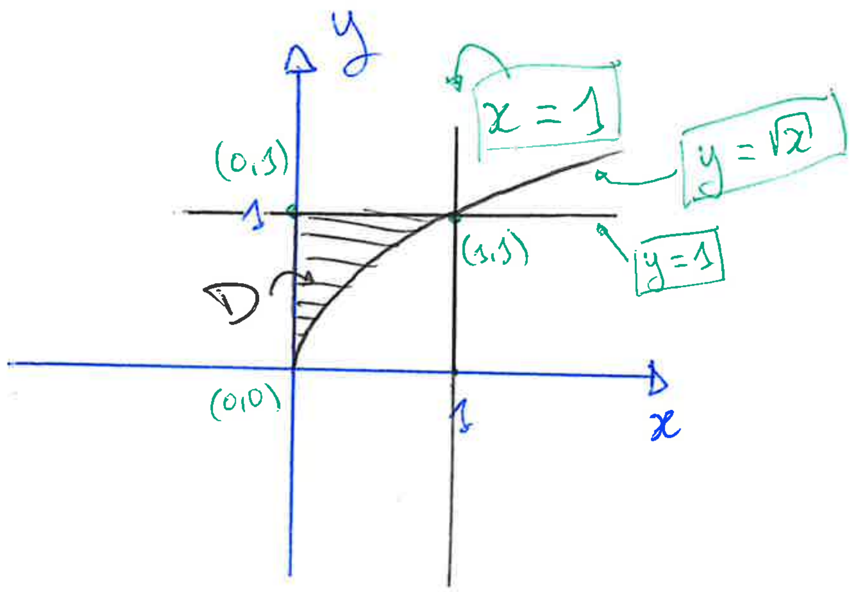
$$\int_{C_2} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \phi(2,1,5) - \phi(1,-1,2)$$

$$= \left(2.1.5 + \frac{e^{1^2}}{2}\right) - \left(1.(-1).2 + \frac{e^{(-1)^2}}{2}\right)$$

$$= 10 + \cancel{\frac{e}{2}} + 2 - \cancel{\frac{e}{2}} = \boxed{12}.$$

Exercise 2:

a) i)



ii) D as an x-simple domain:

$(\sqrt{x} \leq y \Rightarrow x \leq y^2)$

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ 0 \leq x \leq y^2\}$$

iii)
$$\int_0^1 \int_0^{y^2} \cos\left(\frac{\pi y^3}{2}\right) dx\, dy = \int_0^1 y^2 \cos\left(\frac{\pi y^3}{2}\right) dy$$

$$= \frac{2}{3\pi} \int_0^1 \frac{3\pi}{2} y^2 \cos\left(\frac{\pi y^3}{2}\right) dy$$

$$= \frac{2}{3\pi} \left[\sin\left(\frac{\pi y^3}{2}\right)\right]_0^1$$

$$= \boxed{\frac{2}{3\pi}}.$$

b) i) Since P is perpendicular to the line with direction vector $\underset{\sim}{v} = (0,-1,2)$, then $\underset{\sim}{v}$ is normal to P. Also, the point $(3,-2,0) \in P$, so:

$\mathcal{P}: -y + 2z = 0\cdot 3 + (-2)(-1) + 0\cdot 2$

$\Leftrightarrow \boxed{2z - y = 2.}$

ii) The area of $\mathcal{S}$ is given by the surface integral

$$\mathcal{S} = \iint_{\mathcal{S}} dS :$$

Since $\mathcal{S}$ lies on the plane $2z - y = 2$ and its projection onto the xy-plane is D (1-1 projection), then $dS = \boxed{\dfrac{|\underset{\sim}{\nabla} G(x,y,z)|}{\left|\frac{\partial G}{\partial z}\right|} dx dy,}$ $(x,y) \in D$

where $G(x,y,z) = 2z - y$. Hence,

$$dS = \frac{|-\underset{\sim}{j} + 2\underset{\sim}{k}|}{2} dx dy = \frac{\sqrt{1+4}}{2} dx dy = \boxed{\frac{\sqrt{5}}{2} dx dy}$$

$$\Rightarrow \iint_{\mathcal{S}} dS = \iint_D \frac{\sqrt{5}}{2} dx dy = \frac{\sqrt{5}}{2} \int_0^1 \int_0^{y^2} dx dy$$

$$= \frac{\sqrt{5}}{2} \int_0^1 y^2 dy = \frac{\sqrt{5}}{2} \cdot \frac{1}{3} [y^3]_0^1$$

$$= \boxed{\frac{\sqrt{5}}{6}}$$

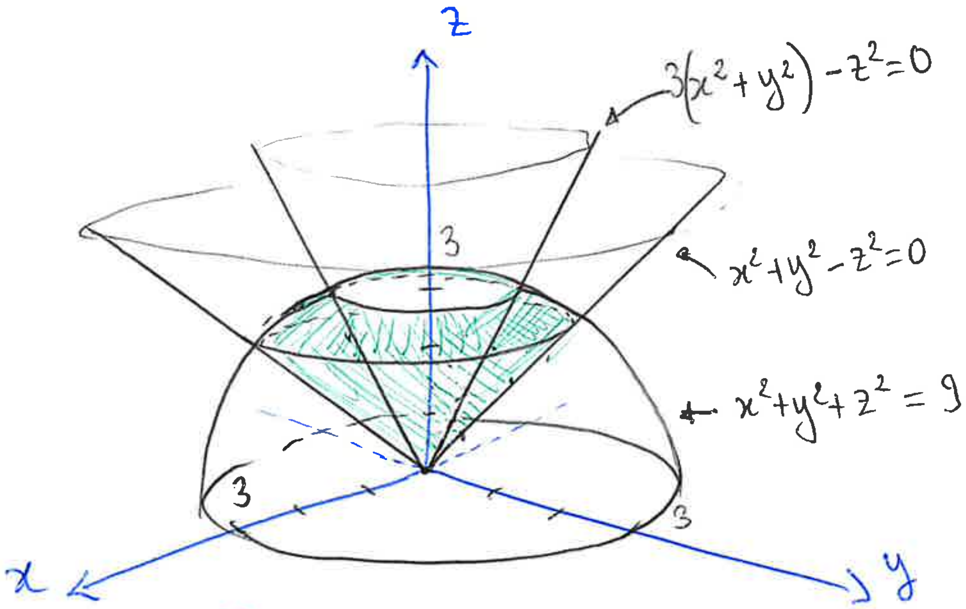
Exercise 3 :

a) i) Since the reflexion $(x,y,z) \mapsto (-x,y,z)$ is a symmetry of T (see picture below), and since

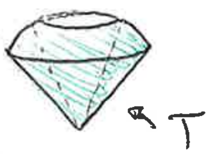
$$f(-x,y,z) = -f(x,y,z) \text{ for every } (x,y,z) \in T \text{, then}$$

$\displaystyle\iiint_{\substack{T\\x>0}} f(x,y,z)\,dV$ and $\displaystyle\iiint_{\substack{T\\x<0}} f(x,y,z)\,dV$ are equal in size but opposite in sign, and so they cancel out when we sum them. And hence,

$$\iiint_T f(x,y,z)\,dV = \cancel{\iiint_{\substack{T\\x>0}} f\,dV + \iiint_{\substack{T\\x<0}} f\,dV} = 0.$$

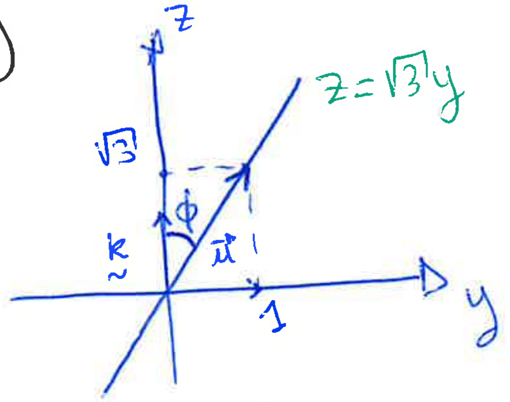


T is the Region in green,



and we see that it is indeed symmetric w.r.t $(x,y,z) \mapsto (-x,y,z)$ (corresponding to symmetry w.r.t the $(y-z)$-plane).

a) ii)



Let $\vec{u} = (1, \sqrt{3})$ be a direction vector of $z = \sqrt{3}y$.

Then, $\vec{u} \cdot \underset{\sim}{k} = |\vec{u}| \, \overbrace{|\underset{\sim}{k}|}^{1} \cos\phi$

$\Rightarrow \quad \sqrt{3} = \sqrt{1+3} \cos\phi$

$\Rightarrow \quad \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\phi = \frac{\pi}{6}}$

b) Since $-4xy$ is odd for the x-variable, $\iiint_T -4xy \, dV \overset{0}{=}$ is zero ~~zero~~ by symmetry of T.

So $\boxed{\iiint_T (3z - 4xy) dV = 3 \iiint_T z \, dV}$

we use spherical coordinates:

$0 \leq R \leq 3 \;, \quad 0 \leq \theta \leq 2\pi \;, \quad \frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}$

angle that the cone $3(x^2+y^2) - z^2 = 0$ makes with positive z-axis that we computed in the previous question.

angle that the cone $x^2+y^2-z^2=0$ makes with positive z-axis

So, $3\iiint_T z\,dV = 3\int_0^{2\pi}\int_0^{3}\int_{\pi/6}^{\pi/4} R\cos\phi \cdot R^2\sin\phi \, d\phi\, dR\, d\theta$

$$= 6\pi \int_0^3 \int_{\pi/6}^{\pi/4} R^3 \cos\phi \sin\phi \, d\phi \, dR$$

$$= 6\pi \int_0^3 R^3 \left[\frac{\sin^2\phi}{2}\right]_{\pi/6}^{\pi/4} dR$$

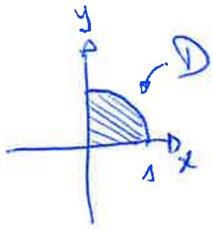
$$= 3\pi \int_0^3 R^3 \left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) dR$$

$$= 3\pi\left(\frac{2}{4} - \frac{1}{4}\right)\left[\frac{R^4}{4}\right]_0^3 = \frac{3\pi}{16}(81-0)$$

$$= \boxed{\frac{243\pi}{16}}.$$

c) We compute $\iint_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS$:

S has a 1-1 projection onto the xy-plane, which is the quarter disc $D = \{x^2+y^2 \leq 1, x>0, y>0\}$



Let $G(x,y,z) = z + y^2 + x^2$, then

$$\hat{\mathbf{N}}\,dS = \pm \frac{(2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})}{1}\,dx\,dy$$

we pick the $(-)$ sign in order to take the normal vector that is pointing downwards (i.e. the vector with negative $\mathbf{k}$ component) because we are asked for the flux downward:

$$\boxed{\hat{\underset{\sim}{N}}\,dS = \left(-2x\,\underset{\sim}{i} - 2y\,\underset{\sim}{j} - \underset{\sim}{k}\right)dx\,dy}$$

$$\iint_{S} \underset{\sim}{F}\cdot\hat{\underset{\sim}{N}}\,dS = \iint_{D} \cancel{-2xy} + \cancel{2yx} + z\,dx\,dy = \iint_{D} (4 - x^2 - y^2)\,dx\,dy$$

polar coordinates: $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 (4-r^2)\,r\,dr\,d\theta = \frac{\pi}{2}\int_0^1 (4r - r^3)\,dr$

$$= \frac{\pi}{2}\left(2\left[r^2\right]_0^1 - \left[\frac{r^4}{4}\right]_0^1\right)$$

$$= \frac{\pi}{2}\left(\frac{8}{4} - \frac{1}{4}\right) = \boxed{\frac{7\pi}{8}}.$$

d) The gradient of f at P gives (a vector in) the direction of the maximum increase of f at P:

$$\underset{\sim}{\nabla} f(x,y) = -2x\,\underset{\sim}{i} - 2y\,\underset{\sim}{j} \Rightarrow \underset{\sim}{\nabla} f(-1,-1) = (2,2).$$

then, a unit vector in this direction is:

$$\hat{\underset{\sim}{v}} = \frac{(2,2)}{|(2,2)|} = \boxed{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}.$$

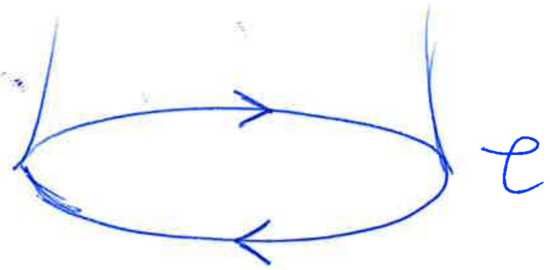
Exercise 4:

a) $\nabla \cdot \underset{\sim}{F} = \boxed{z + 1}$.

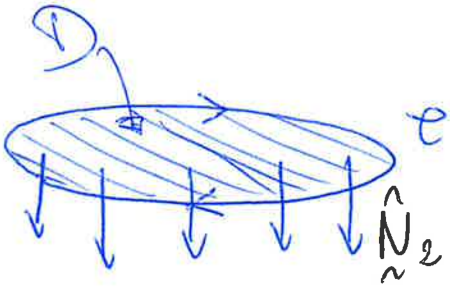
$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz - \frac{y^3 \cos z}{3} & \frac{x^3 \cos z}{3} & xy + z \end{vmatrix} = \underset{\sim}{i}\left(x + \frac{x^3 \sin z}{3}\right) - \underset{\sim}{j}\left(y - x - \frac{y^3 \sin z}{3}\right) + \underset{\sim}{k}\left(x^2 \cos z + y^2 \cos z\right)$$

$$= \boxed{\left(x + \frac{x^3 \sin z}{3}\right)\underset{\sim}{i} + \left(x + \frac{y^3 \sin z}{3} - y\right)\underset{\sim}{j} + \left(x^2 + y^2\right)\cos z \, \underset{\sim}{k}}$$

b) i) If we imagine we are walking on the inner side of $\mathcal{S}$ along $\mathcal{C}$, then in order for our left arm to point towards $\mathcal{S}$ we need to walk clockwise:



ii) To get the equality by Stokes' theorem, $\underset{\sim}{\hat{N}}_2$ must be pointing in the direction that induces the clockwise orientation on $\mathcal{C}$ (when seen from above), so $\underset{\sim}{\hat{N}}_2 = -\underset{\sim}{k}$:



iii) By Stokes' theorem, we have

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D (\nabla \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}_2 \, dS \quad , \quad \boxed{\begin{matrix} \hat{\underset{\sim}{N}}_2 = -\underset{\sim}{k} \\ z = 0 \end{matrix}}$$

because D is on the xy-plane

$$= \iint_D -(x^2+y^2)\cos z \Big|_{z=0} \, dx\,dy$$

$$= -\iint_D (x^2+y^2)\cos(0)\, dx\,dy = -\iint_D (x^2+y^2)\,dx\,dy$$

polar coordinates:

$$= -\int_0^{2\pi}\int_0^{a} r^3\,dr\,d\theta = \boxed{-\frac{2\pi a^4}{4}}.$$